The 44th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

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COMPREHENSIVE EXAMINATION

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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions are not arranged in order of difficulty. For each question, choose the best of the five options labeled A, B, C, D and E. Calculators are NOT permitted.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a paper with: all questions answered correctly earns a score of 250, all questions left blank earns a score of 50, and all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\widehat{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\widehat{AB}$ is the measure of $\widehat{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Analysis
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, Northrop Grumman, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Two opposite sides of a square are increased by 25% and the other two sides are decreased by 60%. What is the percent decrease in the area of the resulting rectangle?

(**A**) 20% (**B**) 30% (**C**) 40% (**D**) 50% (**E**) None of these

Answer: D) 50%

2. Let α be the angle expressed in radians satisfying $\frac{\pi}{2} < \alpha < \pi$ and $\cos^4(\alpha) - \sin^4(\alpha) = \frac{1}{2}$. What is the value of α?

(**A**) $\frac{2\pi}{3}$ (**B**) $\frac{7\pi}{8}$ (**C**) $\frac{5\pi}{6}$ (**D**) $\frac{3\pi}{4}$ (**E**) None of these

Answer C) $\frac{5\pi}{6}$

3. Find the perimeter of a regular polygon with sides of length $16u$ and interior angles measuring $162°$.

(**A**) $288u$ (**B**) $320u$ (**C**) $352u$ (**D**) $384u$ (**E**) None of these

Answer: B) $320u$

4. For all x wherever defined, which expression is equal to

$$1 + \frac{x}{1 - \frac{2}{2+\frac{1}{4x}}}?$$

(**A**) $\frac{1-3x}{1-4x}$ (**B**) $\frac{7}{8}$ (**C**) $1 + \frac{x^2}{8}$ (**D**) $8x^2 + x + 1$ (**E**) None of these

Answer: D) $8x^2 + x + 1$

5. A jar contains \$5.05 made up of d dimes and q quarters. If there are 40 coins, what is the value of q?

(**A**) 7 (**B**) 33 (**C**) 45 (**D**) 10 (**E**) None of these

Answer: A) 7

6. For what values of the real number m, does the equation $(m-3)x^2 - 2x + 2 = 0$ have exactly two distinct real solutions?

(**A**) $\left(-\infty, \frac{7}{2}\right)$ (**B**) $\left(-\infty, \frac{7}{2}\right]$ (**C**) $\left(\frac{7}{2}, \infty\right)$ (**D**) $\left[\frac{7}{2}, \infty\right)$ (**E**) None of these

Answer: A) $(-\infty, \frac{7}{2})$

7. Consider the function $P(x) = ax^2 + bx + c$. It is known that $x = 1$ and $x = 3$ are two zeros and that $P(2) = 2$. Find $P(0)$.

(**A**) -6 (**B**) -4 (**C**) 4 (**D**) 6 (**E**) None of these

Answer: A) -6

8. Suppose $\sin(\theta) = -0.8$ and $\pi < \theta < \frac{3\pi}{2}$. What is the value of $\tan(\theta)$?

(**A**) $-\frac{2}{3}$ (**B**) $-\frac{3}{4}$ (**C**) $\frac{4}{3}$ (**D**) $\frac{3}{2}$ (**E**) None of these

Answer C) $\frac{4}{3}$

9. Determine the number of solutions to the equation $x = \tan(2x)$ on the interval $[-3\pi, 3\pi]$.

(**A**) 5 (**B**) 11 (**C**) 6 (**D**) 10 (**E**) None of these

Answer B) 11

10. If the radius of a sphere is doubled, the volume is made:

(**A**) 4 times as great (**B**) 6 times as great (**C**) 8 times as great (**D**) 12 times as great (**E**) None of these

Answer: C) 8 times as great

11. What is the equation of the perpendicular bisector of the segment $\overline{AB}$ with endpoints $A(-12, 15)$ and $B(4, -3)$?

(**A**) $y = \frac{8}{9}x + \frac{86}{9}$ (**B**) $y = \frac{9}{8}x - \frac{86}{9}$ (**C**) $y = -\frac{8}{9}x - \frac{86}{9}$ (**D**) $y = -\frac{9}{8}x + \frac{86}{9}$ (**E**) None of these

Answer: A) $y = \frac{8}{9}x + \frac{86}{9}$

12. Let $p(x)$ be a quadratic with leading coefficient one with $p(3) = 0$ and $p(4) = 2$. Determine $p(2)$.

(**A**) -2 (**B**) -1 (**C**) 0 (**D**) 1 (**E**) 5

Answer: C) 0

13. Let f be a function satisfying $(x + y)f(x) = xf(x + y)$ for all positive real numbers x and y. If $f(2025) = 135$, compute $f(15)$.

(**A**) 1 (**B**) 15 (**C**) 45 (**D**) 135 (**E**) None of these

$f(n) = nf(1)$ since $xf(nx) = xf(x + (n-1)x) = (x + (n-1)x)f(x) = nxf(x)$, $x = 1$. **Answer:** A) 1

14. Which of the following is equivalent to the expression

$$\left(\frac{8\left(\frac{a^4}{b^3}\right)^2}{\frac{125}{ab^3}}\right)^{-\frac{1}{3}}$$

for all nonzero values of a and b?

(**A**) $\frac{2}{5}a^3b$ (**B**) $\frac{2}{5}a^{-3}b$ (**C**) $\frac{5}{2}a^{-3}b$ (**D**) $\frac{5}{2}a^{-3}b^{-1}$ (**E**) None of these

Answer: C) $\frac{5}{2}a^{-3}b$

15. Determine the number of pairs (u, v) of positive integers such that $u^{20}v^{25} = 2^{2025}$.

(**A**) 20 (**B**) 25 (**C**) 45 (**D**) 50 (**E**) None of these

Answer: A) 20

16. What is the area of the triangle that has sides of length, 5, 10, and 13?

(**A**) $2\sqrt{14}$ (**B**) $6\sqrt{14}$ (**C**) $18\sqrt{14}$ (**D**) $24\sqrt{14}$ (**E**) None of these

Answer: B) $6\sqrt{14}$

17. A chord is perpendicular to a diameter of a circle at a point which divides the diameter into segments having the ratio $1 : 3$. In what ratio does the chord divide the circumference?

(**A**) $2 : 3$ (**B**) $3 : 4$ (**C**) $1 : 2$ (**D**) $1 : 4$ (**E**) None of these

Answer: C) 1 : 2

18. Determine the surface area of a sphere whose volume is 288π.

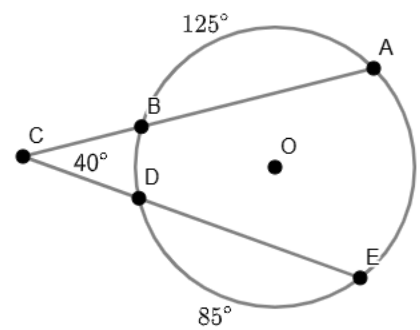
(**A**) 120π (**B**) 144π (**C**) 168π (**D**) 192π (**E**) None of these

Answer: B) 144π

19. In the circle shown, $m\angle C = 40°$, $m\widehat{AB} = 125°$ and $m\widehat{ED} = 85°$. Determine $m\widehat{AE}$.

(**A**) $100°$ (**B**) $105°$ (**C**) $110°$

(**D**) $115°$ (**E**) None of these



Answer: D) $115°$

20. If $g(x) = 3 + x + xe^x$, find $g^{-1}(\ln(8e^3))$.

(**A**) $\ln(2)$ (**B**) $-\ln(2)$ (**C**) $\ln(8)$ (**D**) $-\ln(8)$ (**E**) None of these

Answer: A) $\ln(2)$

21. Let $g(x) = ax^2+bx+c$ be a polynomial of degree 2 with nonzero positive integer coefficients. Suppose that $g(n)$ is divisible by 5 for every integer n. What is the smallest possible value of abc?

(**A**) 5 (**B**) 25 (**C**) 125 (**D**) 625 (**E**) None of these

Answer: C) 125

22. How many distinct real roots does the polynomial $x^4 - 3x^3 + 4x^2 - 3x + 1$ have?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) None of these

Answer: A) 1

23. Which of the following expressions has the largest value for $0 < x < 1$?

(**A**) $\frac{1}{x}$ (**B**) x^2 (**C**) $\frac{1}{1+x}$ (**D**) $\frac{1}{\sqrt{x}}$ (**E**) $\sqrt{x}$

Answer: A) $\frac{1}{x}$

24. A fenced-in rectangular plot of land contains a rectangular pool surrounded by a lawn. If the pool is 40 meters by 16 meters and is exactly 2 meters from the fence at all points, find the area of the lawn.

(**A**) 214 m^2 (**B**) 240 m^2 (**C**) 266 m^2 (**D**) 282 m^2 (**E**) None of these

Answer: B) 240 m^2

25. If the perimeters of a square and an equilateral triangle both equal 3, then find the positive difference of the area of the square and the area of the triangle?

(**A**) $\frac{9-4\sqrt{3}}{16}$ (**B**) $\frac{6-2\sqrt{3}}{16}$ (**C**) $\frac{8\sqrt{3}-11}{16}$ (**D**) $\frac{9-2\sqrt{3}}{16}$ (**E**) None of these

Answer: A) $\frac{9-4\sqrt{3}}{16}$

26. Find the distance between the three-dimensional points $(6, 4, -3)$ and $(2, -8, 3)$.

(**A**) 14 (**B**) $15\sqrt{2}$ (**C**) 16 (**D**) $12\sqrt{3}$ (**E**) None of these

Answer: A) 14

27. Two equilateral triangles sharing an edge have a combined area of π. What is the square of the length of their shared edge?

(**A**) $\frac{\pi\sqrt{3}}{3}$ (**B**) $\frac{2\pi\sqrt{3}}{3}$ (**C**) $\frac{4\pi\sqrt{3}}{3}$ (**D**) $\frac{5\pi\sqrt{3}}{3}$ (**E**) None of these

Answer: B) $\frac{2\pi\sqrt{3}}{3}$

28. The equation $ax^2 + bx + 1 = 0$ has solutions $x = 1$ and $x = 2$. What are the values of a and b?

(**A**) $a = \frac{1}{2}, b = -\frac{3}{2}$ (**B**) $a = -\frac{1}{2}, b = \frac{3}{2}$ (**C**) $a = \frac{1}{2}, b = \frac{3}{2}$
(**D**) $a = -\frac{1}{2}, b = -\frac{3}{2}$ (**E**) None of these

Answer: A) $a = \frac{1}{2}$ and $b = -\frac{3}{2}$

29. Define the sequence a_n by $a_0 = 1$ and $a_n = 2a_{n-1} + \frac{1-(-1)^n}{2}$ for $n \geq 1$. Find the smallest N such that $a_N \geq 2025$.

(**A**) 7 (**B**) 11 (**C**) 13 (**D**) 17 (**E**) None of these

Answer: B) 11

30. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be an invertible matrix such that $A^2 = 3A - 2$ and $a_{12} = a_{21} = \frac{1}{2}$. Compute the sum $a_{11} + a_{12} + a_{21} + a_{22}$.

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) None of these

Answer: D) 4

31. For any positive integer n, the nth triangle number is $t_n = 1 + 2 + 3 + \cdots + n$. The number 2025 is the square of a triangle number t_N, for some N. Compute the difference $t_{N+1}^2 - t_N^2$.

(**A**) 1000 (**B**) 1331 (**C**) 3375 (**D**) 2025^2 (**E**) None of these

Answer: A) 1000

32. Find the largest positive integer K such that $\dfrac{n^4 - n^2}{K}$ is always an integer when n is an integer.

(**A**) 4 (**B**) 6 (**C**) 12 (**D**) 24 (**E**) None of these

Answer: C) 12

33. Let $AB = 2\sqrt{3}$ and $CD = 2$ be two chords on a circle with center O. The ratio of the distance from O to $\overline{AB}$ and the distance from O to $\overline{CD}$ is $1 : \sqrt{3}$. Find the circumference of the circle.

(**A**) π (**B**) $\sqrt{2}\pi$ (**C**) 2π (**D**) $2\sqrt{2}\pi$ (**E**) None of these

Answer: E) None of these

34. The sides a, b, and c of a triangle satisfy $\sqrt{a}+\sqrt{b}=\sqrt{c}$. Which of the following best describes the triangle?

(**A**) Acute (**B**) Scalene (**C**) Isosceles (**D**) Equilateral (**E**) None of these

Answer: E) None of these

35. Consider the complex numbers $z_1 = 2+3i$ and $z_2 = 4+5i$. Recall that the modulus of a complex number $a+bi$ is $|a+bi| = \sqrt{a^2+b^2}$. If $a = |z_1|$ and $b = |z_2|$, compute $|a+bi|$

(**A**) $\sqrt{\sqrt{5}+\sqrt{9}}$ (**B**) $4\sqrt{7}$ (**C**) $3\sqrt{6}$ (**D**) $\sqrt{\sqrt{13}+\sqrt{41}}$ (**E**) None of these

Answer: C) $3\sqrt{6}$

36. Consider all polynomials of the form

$$(1-t^{a_1})(1-t^{a_2})\cdots(1-t^{a_r})$$

where $r \geq 1$ is an integer and $a_1, a_2, \ldots, a_r$ are positive integers adding up to 3. Find the degree of the least common multiple of these polynomials.

(**A**) 6 (**B**) 9 (**C**) 12 (**D**) 15 (**E**) None of these

Answer: A) 6

37. Consider the piecewise function

$$f(x) = \begin{cases} -2x+1 & x \leq -1, \\ x+4 & -1 < x \leq 0, \\ -2x+4 & 0 < x \leq 1, \\ 2x & x > 1. \end{cases}$$

This can be written in the form

$$f(x) = A|x+1| + B|x| + C|x-1| + D$$

for some real numbers A, B, C and D. Determine $A^2+B^2+C^2+D^2$.

(**A**) $\frac{15}{4}$ (**B**) $\frac{65}{9}$ (**C**) $\frac{35}{4}$ (**D**) $\frac{79}{9}$ (**E**) None of these

Answer: C) $\frac{35}{4}$

38. Let n be an integer. Which of the following is equal to $3^{n+1}+3^{n-2}-\frac{1}{3^{1-n}}$?

(**A**) $3^{n-2}\times 25$ (**B**) 3^{n-1} (**C**) $3^{n-1}\times 25$ (**D**) 3^{2n-1} (**E**) None of these

Answer: A) $3^{n-2}\times 25$

39. Suppose $C = \cos(\theta)$, where $0 \leq \theta \leq \dfrac{\pi}{2}$. Which of the following is equal to $\sqrt{\dfrac{\sqrt{\frac{C+1}{2}}+1}{2}}$?

(**A**) $\sin\left(\frac{\theta}{2}\right)$ (**B**) $\sin\left(\frac{\theta}{4}\right)$ (**C**) $\cos\left(\frac{\theta}{2}\right)$ (**D**) $\cos\left(\frac{\theta}{4}\right)$ (**E**) None of these

Answer D) $\cos\left(\frac{x}{16}\right)$

40. A circle is inscribed in an isosceles trapezoid which has bases measuring 8 and 18. Find the area between the circle and isosceles trapezoid.

(**A**) $144 - 28\pi$ (**B**) $121 - 16\pi$ (**C**) $172 - 49\pi$ (**D**) $156 - 36\pi$ (**E**) None of these

Answer: D) $156 - 36\pi$

41. Which of the following expressions is equal to $\cos(3\pi - x) + \cos(\frac{\pi}{2} - x) + \sin(-\frac{3\pi}{2} - x)$ for all $x \in \mathbb{R}$?

(**A**) $2\cos(x) - \sin(x)$ (**B**) $-2\cos(x) - \sin(x)$ (**C**) $\sin(x)$ (**D**) $-\sin(x)$ (**E**) None of these

Answer C) $\sin(x)$

42. Let m and n be two non-negative integers. If $m! \cdot n! = 10!$, what is the largest possible value for the product $m \cdot n$?

(**A**) 0 (**B**) 10 (**C**) 42 (**D**) 3628800 (**E**) None of these

Answer: C) 42

43. Let a and b be two positive numbers such that $a^2 \geq b$. Which of the following expressions is equal to

$$\sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}\,?$$

(**A**) $\sqrt{\frac{a+b}{2}}$ (**B**) $\sqrt{a-\sqrt{b}}$ (**C**) $\sqrt{a+\sqrt{b}}$ (**D**) $\sqrt{\sqrt{a}+b}$ (**E**) None of these

Answer: C) $\sqrt{a+\sqrt{b}}$

44. A student has taken three exams. If the student gets the same score on the fourth exam as the first, then her average on all four would be 10. If it matches the second exam instead, then her average would be 12. If it matches the third, her average would be 14. What were the first three exam scores?

(**A**) 2, 10, and 26 (**B**) 3, 11, and 23 (**C**) 4, 12, and 20 (**D**) 5, 9, and 21 (**E**) None of these

Answer: C) 4, 12, and 20

45. Let (x, y) be a solution to the following system of two equations.

$$\begin{cases} 4x + y = & 2 \\ 2x - 3y = & 8 \end{cases}$$

Which of the following statements is true?

(**A**) $xy = 2$ (**B**) $x - y = -3$ (**C**) $x^2 + y^2 = 5$

(**D**) The system has no solution (**E**) None of these

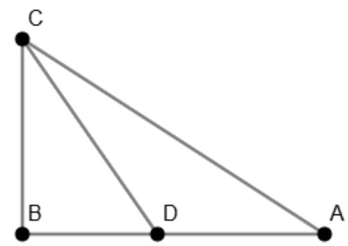
Answer: C) $x^2 + y^2 = 5$

46. Find the mean of the y-values of the the points of intersection between $y = \sin(x) + \cos(x)$ and $y = \sin^3(x) + \cos^3(x)$ such that $x \in [0, \pi]$

(**A**) 0 (**B**) $\frac{1}{4}$ (**C**) $\frac{1}{3}$ (**D**) $\frac{1}{2}$ (**E**) None of these

Answer B) $\frac{1}{4}$

47. Suppose $m\angle ABC = 90°$, $m\angle CDB = 45°$, $m\angle CAB = 30°$, and $AD = 2$. Then BC equals:



(**A**) $\sqrt{3}+1$ (**B**) $2\sqrt{3}$ (**C**) $2\sqrt{3}+1$

(**D**) $\sqrt{3}+2$ (**E**) None of these

Answer: A) $\sqrt{3}+1$

48. Compute $\displaystyle\sum_{n=46}^{2025} \log_2\left(\log_{n-1}(n)\right)$.

(**A**) 1 (**B**) 2 (**C**) 15 (**D**) 45 (**E**) None of these

Answer: A) 1

49. Let $\mathbb{R}_{>0}$ be the set of positive real numbers and let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be the function that satisfies $f(1) = \frac{1}{2025}$ and

$$f(x)f(yf(x)) = f(x+y)$$

for all $x, y \in \mathbb{R}_{>0}$. Compute $f(2026)$.

(**A**) $\dfrac{1}{2026}$ (**B**) $\dfrac{2}{2025}$ (**C**) $\dfrac{1}{2025 \cdot 2026}$ (**D**) $\dfrac{1}{2025^2}$ (**E**) None of these

Answer: D) $\dfrac{1}{2025^2}$

50. Let $f(x) = x^3 + x^2 - 4x - 4$ and define the function $g(x) = f(x) - \sqrt{(f(x))^2}$. For which of the following intervals is $g(x)$ equal to 0 for all values of x in the interval?

(**A**) $[-1, 2]$ (**B**) $[2, 4]$ (**C**) $[-2, 2]$ (**D**) All of the above (**E**) None of these

$g(x) = 0$ whenever $f(x) \geq 0$. Therefore, $g(x) = 0$ on $[-2, -1] \cup [2, \infty)$ and is negative otherwise. **Answer:** B) $[2, 4]$