The 44th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

Written Round: February 22, 2025 at Regional Testing Sites

Ciphering Round: April 12, 2025 at University of Alabama at Birmingham

ALGEBRA II EXAMINATION

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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions are not arranged in order of difficulty. For each question, choose the best of the five options labeled A, B, C, D and E. Calculators are NOT permitted.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a paper with: all questions answered correctly earns a score of 250, all questions left blank earns a score of 50, and all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\widehat{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\widehat{AB}$ is the measure of $\widehat{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Printing by The University of Alabama at Birmingham

Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Analysis
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, Northrop Grumman, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Let m and n be two non-negative integers. If $m! \cdot n! = 10!$, what is the largest possible value for the product $m \cdot n$?

(**A**) 0 (**B**) 10 (**C**) 42 (**D**) 3628800 (**E**) None of these

2. A toy bank contains quarters, dimes, and nickels, having a total value of \$4.15. The number of nickels is one-third the number of dimes, and the number of quarters is 7 more than the number of nickels. How many dimes are there in the bank?

(**A**) 4 (**B**) 12 (**C**) 18 (**D**) 6 (**E**) None of these

3. Let Z be a nonzero complex number. How many solutions z does the following system of equations have?

$$\begin{cases} z^2 = Z \\ Z^2 = z \end{cases}$$

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) None of these

4. For what values of the real number m, does the equation $(m-3)x^2 - 2x + 2 = 0$ have exactly two distinct real solutions?

(**A**) $(-\infty, \frac{7}{2})$ (**B**) $(-\infty, \frac{7}{2}]$ (**C**) $(\frac{7}{2}, \infty)$ (**D**) $[\frac{7}{2}, \infty)$ (**E**) None of these

5. Consider the following system of two equations in the unknowns x and y.

$$\begin{cases} 3x + 2y = 5 \\ 2x + 3y = 10 \end{cases}$$

What is the value of $x^2 - y^2$?

(**A**) -25 (**B**) -15 (**C**) 9 (**D**) 25 (**E**) None of these

6. Let a, b and c be positive numbers and set $R = \frac{a^2 b}{c}$. Which of the following is always true for any value of a, b and c.

(**A**) $\ln(Rc) = \ln(2a) + \ln b$ (**B**) $\ln R + \ln c = \ln(2a + b)$

(**C**) $\ln R - \ln b = 2\ln a - \ln c$ (**D**) $\ln(R - b) = 2\ln(a - c)$ (**E**) None of these

7. Which of the following expressions is equal to

$$\frac{1}{(x-y)(y-z)} - \frac{1}{(y-z)(z-x)} - \frac{1}{(z-x)(x-y)}?$$

(**A**) $\frac{y+z}{(x-y)(y-z)(z-x)}$ (**B**) $\frac{2}{(x-y)(y-z)}$ (**C**) 0 (**D**) $\frac{x+z}{(x-y)(y-z)(z-x)}$ (**E**) None of these

8. Which of the following is equivalent to the expression

$$\left(\frac{8\left(\frac{a^4}{b^3}\right)^2}{\frac{125}{ab^3}} \right)^{-\frac{1}{3}}$$

for all nonzero values of a and b?

(**A**) $\frac{2}{5}a^3 b$ (**B**) $\frac{2}{5}a^{-3} b$ (**C**) $\frac{5}{2}a^{-3} b$ (**D**) $\frac{5}{2}a^{-3} b^{-1}$ (**E**) None of these

9. A jar contains \$5.05 made up of d dimes and q quarters. If there are 40 coins, what is the value of q?

(A) 7 **(B)** 33 **(C)** 45 **(D)** 10 **(E)** None of these

10. For all x wherever defined, which expression is equal to

$$1+\frac{x}{1-\frac{2}{2+\frac{1}{4x}}}?$$

(A) $\frac{1-3x}{1-4x}$ **(B)** $\frac{7}{8}$ **(C)** $1+\frac{x^2}{8}$ **(D)** $8x^2+x+1$ **(E)** None of these

11. Let a and b be two positive numbers such that $a^2 \geq b$. Which of the following expressions is equal to

$$\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}?$$

(A) $\sqrt{\frac{a+b}{2}}$ **(B)** $\sqrt{a-\sqrt{b}}$ **(C)** $\sqrt{a+\sqrt{b}}$ **(D)** $\sqrt{\sqrt{a}+b}$ **(E)** None of these

12. The equation $ax^2+bx+1=0$ has solutions $x=1$ and $x=2$. What are the values of a and b?

(A) $a=\frac{1}{2}, b=-\frac{3}{2}$ **(B)** $a=-\frac{1}{2}, b=\frac{3}{2}$ **(C)** $a=\frac{1}{2}, b=\frac{3}{2}$

(D) $a=-\frac{1}{2}, b=-\frac{3}{2}$ **(E)** None of these

13. Let n be an integer. Which of the following is equal to $3^{n+1}+3^{n-2}-\frac{1}{3^{1-n}}$?

(A) $3^{n-2}\times 25$ **(B)** 3^{n-1} **(C)** $3^{n-1}\times 25$ **(D)** 3^{2n-1} **(E)** None of these

14. Let (x, y) be a solution to the following system of two equations.

$$\begin{cases} 4x+y= & 2 \\ 2x-3y= & 8 \end{cases}$$

Which of the following statements is true?

(A) $xy=2$ **(B)** $x-y=-3$ **(C)** $x^2+y^2=5$

(D) The system has no solution **(E)** None of these

15. Consider the function $P(x)=ax^2+bx+c$. It is known that $x=1$ and $x=3$ are two zeros and that $P(2)=2$. Find $P(0)$.

(A) -6 **(B)** -4 **(C)** 4 **(D)** 6 **(E)** None of these

16. A grandmother has decided to reward her grandchildren with money based on their school performance. The amount she gives is proportional to the difference between their worst result (in %) and 50%. She gave \$15 to Eric, whose worst result was 95%. How much did she give to Rosine, whose worst result was 86%?

(A) \$10 **(B)** \$11 **(C)** \$12 **(D)** \$13 **(E)** None of these

17. Which of the following expressions has the largest value for $0<x<1$?

(A) $\frac{1}{x}$ **(B)** x^2 **(C)** $\frac{1}{1+x}$ **(D)** $\frac{1}{\sqrt{x}}$ **(E)** $\sqrt{x}$

18. A student has taken three exams during a semester. If the student gets the same score on the fourth exam as the first exam, then her average on all four exams would be 10. If it matches the second exam instead, then her average would be 12. If it matches the third exam score, her average would be 14. What were the scores on the first three exams, in chronological order?

(**A**) 2, 10, and 26 (**B**) 3, 11, and 23 (**C**) 4, 12, and 20 (**D**) 5, 9, and 21 (**E**) None of these

19. Let a be a real number and consider the equation $(a^2+1)x^2 + a^2x - 1 = 0$. For any value of a, which of the following statements is true for the equation?

(**A**) It has no real solution (**B**) It has at most one real solution

(**C**) It has exactly one real solution (**D**) It has two distinct real solutions (**E**) None of these

20. The sequence of rational numbers $a_1, a_2, \cdots, a_N$ has following property: for each integer k, $1 \le k \le 10$,

$$\#\left\{1 \le i \le N \ : a_i = \frac{1}{k}\right\} = k^3.$$

Assuming that for each i, $1 \le i \le N$, $a_i \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10}\}$, compute $\sum_{i=1}^{N} a_i$.

(**A**) 45 (**B**) 2025 (**C**) 385 (**D**) $\frac{1}{2025}$ (**E**) None of these

21. We call a function f defined on $[0,1]$ a contraction if there is a real number $0 \le \alpha < 1$ where f satisfies

$$|f(x) - f(y)| \le \alpha|x - y|$$

for any $x, y \in [0,1]$. Which of the following functions is not a contraction?

(**A**) $a(x) = \frac{1}{2}\sqrt{x}$ (**B**) $b(x) = \frac{1}{3}x$ (**C**) $c(x) = \frac{1}{4}x^2$ (**D**) $d(x) = \frac{1}{5}x^3$ (**E**) None of these

22. Define the function f on the real numbers by $f(x) = \sqrt{x^4 + x^2} - x^2$. Which of the following values of M satisfy the inequality $\frac{M}{2} \le f(x) \le M$ for all real numbers $x \ge 1$?

(**A**) 1 (**B**) $\frac{1}{2}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{1}{8}$ (**E**) None of these

23. Suppose f and g are functions defined on the real numbers such that f is odd and invertible and g is even. Which of the following statements must be true?

(**A**) $a(x) = (f^{-1} \circ g)(x)$ is odd (**B**) $b(x) = f^{-1}(x) + g(x)$ is odd

(**C**) $c(x) = f^{-1}(x) \cdot g(x)$ is odd (**D**) $d(x) = (f \circ g \circ f^{-1})$ is odd (**E**) None of these

24. How many complex numbers z have the property that $z^{12} = -1$ and $z^{16} = 1$?

(**A**) 48 (**B**) 12 (**C**) 8 (**D**) 4 (**E**) None of these

25. How many distinct real roots does the polynomial $x^4 - 3x^3 + 4x^2 - 3x + 1$ have?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) None of these

26. Consider the piecewise function

$$f(x) = \begin{cases} -2x+1 & x \le -1, \\ x+4 & -1 < x \le 0, \\ -2x+4 & 0 < x \le 1, \\ 2x & x > 1. \end{cases}$$

This can be written in the form

$$f(x) = A|x+1| + B|x| + C|x-1| + D$$

for some real numbers A, B, C and D. Determine $A^2 + B^2 + C^2 + D^2$.

(**A**) $\frac{15}{4}$ (**B**) $\frac{65}{9}$ (**C**) $\frac{35}{4}$ (**D**) $\frac{79}{9}$ (**E**) None of these

27. Let f be a function satisfying $(x+y)f(x) = xf(x+y)$ for all positive real numbers x and y. If $f(2025) = 135$, compute $f(15)$.

(**A**) 1 (**B**) 15 (**C**) 45 (**D**) 135 (**E**) None of these

28. Define the sequence a_n by $a_0 = 1$ and $a_n = 2a_{n-1} + \frac{1-(-1)^n}{2}$ for $n \ge 1$. Find the smallest N such that $a_N \ge 2025$.

(**A**) 7 (**B**) 11 (**C**) 13 (**D**) 17 (**E**) None of these

29. Let $p(x)$ be a quadratic polynomial with leading coefficient one with $p(3) = 0$ and $p(4) = 2$. Determine $p(2)$.

(**A**) -2 (**B**) -1 (**C**) 0 (**D**) 1 (**E**) 5

30. The Heaviside function H is defined by

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x \ge 0. \end{cases}$$

Consider the family of functions $f_{a,b}(x) = aH(x) + bH(-x)$. As a, b range over the integers $\{-5, -4, \dots, 4, 5\}$, how many distinct functions $f_{a,b}$ are there?

(**A**) 121 (**B**) 110 (**C**) 100 (**D**) 90 (**E**) None of these

31. Compute

$$\sum_{n=46}^{2025} \log_2\left(\log_{n-1}(n)\right)$$

(**A**) 1 (**B**) 2 (**C**) 15 (**D**) 45 (**E**) None of these

32. If $g(x) = 3 + x + xe^x$, find $g^{-1}(\ln(8e^3))$.

(**A**) $\ln(2)$ (**B**) $-\ln(2)$ (**C**) $\ln(8)$ (**D**) $-\ln(8)$ (**E**) None of these

33. Let S be the set of coordinate pairs in the plane (x, y) that satisfy the system of equations

$$\max\{x, 2y\} = 1,$$

$$\max\{x, y^2\} = x.$$

In the plane, S consists of two line segments which meet at a right angle. Find the length of the shorter line segment.

(**A**) $\frac{1}{2}$ (**B**) $\frac{2}{3}$ (**C**) $\frac{3}{4}$ (**D**) $\frac{3}{2}$ (**E**) None of these

34. Consider all polynomials of the form

$$(1 - t^{a_1})(1 - t^{a_2}) \cdots (1 - t^{a_r})$$

where $r \geq 1$ is an integer and $a_1, a_2, \ldots, a_r$ are positive integers adding up to 3. Find the degree of the least common multiple of these polynomials.

(**A**) 6 (**B**) 9 (**C**) 12 (**D**) 15 (**E**) None of these

35. Let $\mathbb{R}_{>0}$ be the set of positive real numbers and let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be the function that satisfies $f(1) = \frac{1}{2025}$ and

$$f(x)f(yf(x)) = f(x + y)$$

for all $x, y \in \mathbb{R}_{>0}$. Compute $f(2026)$.

(**A**) $\frac{1}{2026}$ (**B**) $\frac{2}{2025}$ (**C**) $\frac{1}{2025 \cdot 2026}$ (**D**) $\frac{1}{2025^2}$ (**E**) None of these

36. Suppose $f(x)$ is the degree 5 polynomial such that $f(n) = 0$ for $n = 0, 1, 2, 3, 4$ and $f(5) = 1$. What is the coefficient of x^4 in $f(x)$?

(**A**) $\frac{1}{5!}$ (**B**) $-\frac{1}{4!}$ (**C**) $-\frac{1}{12}$ (**D**) $\frac{5}{24}$ (**E**) None of these

37. Let $g(x) = ax^2 + bx + c$ be a polynomial of degree 2 with nonzero positive integer coefficients. Suppose that $g(n)$ is divisible by 5 for every integer n. What is the smallest possible value of abc?

(**A**) 5 (**B**) 25 (**C**) 125 (**D**) 625 (**E**) None of these

38. A function f defined over the real numbers has property (TR) if, for any constant $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation (or both). Which of the following functions does **not** have property (TR)?

(**A**) $a(x) = x^2$ (**B**) $b(x) = 2x - 3$ (**C**) $c(x) = -2x + 3$ (**D**) $d(x) = e^{3x}$ (**E**) None of these

39. For any positive integer n, the nth triangle number is $t_n = 1 + 2 + 3 + \cdots + n$. The number 2025 is the square of a triangle number t_N, for some N. Compute the difference $t_{N+1}^2 - t_N^2$.

(**A**) 1000 (**B**) 1331 (**C**) 3375 (**D**) 2025^2 (**E**) None of these

40. Let $f(x) = x^3 + x^2 - 4x - 4$ and define the function $g(x) = f(x) - \sqrt{(f(x))^2}$. For which of the following intervals is $g(x)$ equal to 0 for all values of x in the interval?

(**A**) $[-1, 2]$ (**B**) $[2, 4]$ (**C**) $[-2, 2]$ (**D**) All of the above (**E**) None of these

41. Suppose a is a positive integer such that

$$\frac{1+\sqrt{5}}{2} = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{\ddots}}}.$$

What is the value of a?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) None of these

42. Consider the complex numbers $z_1 = 2+3i$ and $z_2 = 4+5i$. Recall that the modulus of a complex number $a+bi$ is $|a+bi| = \sqrt{a^2+b^2}$. If $a = |z_1|$ and $b = |z_2|$, compute $|a+bi|$

(**A**) $\sqrt{\sqrt{5}+\sqrt{9}}$ (**B**) $4\sqrt{7}$ (**C**) $3\sqrt{6}$ (**D**) $\sqrt{\sqrt{13}+\sqrt{41}}$ (**E**) None of these

43. Suppose d is an integer factor of 2^9+1 such that $d < 2^9+1$. What is the largest possible value of d?

(**A**) 29 (**B**) 9 (**C**) 91 (**D**) 19 (**E**) None of these

44. Find the smallest positive integer value of n such that $\dfrac{n!}{2025}$ is an integer.

(**A**) 11 (**B**) 12 (**C**) 15 (**D**) 45 (**E**) None of these

45. Suppose the polynomial $x^2 - 2x - c$ has integer roots, where $2000 < c < 2100$. Find its largest root.

(**A**) 45 (**B**) 46 (**C**) 47 (**D**) 48 (**E**) None of these

46. Alice, Bobbi, and Chris have \$24 altogether. If Alice's amount of money is doubled (and Bobbi's and Chris's amounts remain unchanged), then they would have \$30 altogether. How much money do Bobbi and Chris have together?

(**A**) 12 (**B**) 15 (**C**) 18 (**D**) 20 (**E**) None of these

47. Define the operation

$$x \circ y = x + y + xy$$

where $x, y \in (-1, \infty)$. Find $y \in (-1, \infty)$ such that $2 \circ y = 0$.

(**A**) $\dfrac{1}{2}$ (**B**) $\dfrac{1}{3}$ (**C**) $-\dfrac{1}{2}$ (**D**) $-\dfrac{2}{3}$ (**E**) None of these

48. Find the smallest solution to the equation

$$\sqrt{15-x^2} = \sqrt{2x^2+6x-9}$$

(**A**) -4 (**B**) -2 (**C**) 2 (**D**) 4 (**E**) None of these

49. Find the sum of all values of x such that

$$|x| - |x+1| - |x+2| = 0$$

(**A**) -4 (**B**) -1 (**C**) 1 (**D**) 4 (**E**) None of these

50. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be an invertible matrix such that $A^2 = 3A - 2$ and $a_{12} = a_{21} = \frac{1}{2}$. Compute the sum $a_{11} + a_{12} + a_{21} + a_{22}$.

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) None of these