The 42nd Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

Written Round: February 25, 2023 at Regional Testing Sites

Ciphering Round: April 15, 2023 at University of North Alabama

COMPREHENSIVE EXAMINATION

Construction of this test directed
by
Ashley Johnson, University of North Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E. A calculator is NOT permitted.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Find the sum of the x-components of the x-intercepts of the line given by $2x + 3y = 5$ and the line perpendicular to it through the point $(1, 1)$.

(**A**) $\frac{7}{6}$ (**B**) $\frac{17}{6}$ (**C**) $\frac{23}{6}$ (**D**) $\frac{25}{6}$ (**E**) None of these

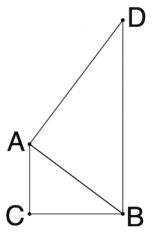
2. If (a, b) is a solution to the system below, find the smallest possible value of $a + b$.

$$\begin{cases} \frac{1}{x^2} - \frac{3}{y^3} = 1 \\ \frac{2}{x^2} - \frac{1}{y^3} = 7 \end{cases}$$

(**A**) $\frac{1}{2}$ (**B**) $\frac{3}{4}$ (**C**) $\frac{3}{2}$ (**D**) $\frac{5}{4}$ (**E**) None of these

3. Suppose $\triangle ABC$ is a right triangle, with legs $AC = 3$ and $BC = 4$. Let D be a point such that $\overleftrightarrow{DB}$ is perpendicular to $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ is perpendicular to $\overleftrightarrow{AB}$. Find the perimeter of $\triangle ABD$.

(**A**) 20 (**B**) 24 (**C**) 27 (**D**) 30 (**E**) None of these



4. The terminal side of an angle given by $\frac{45\pi}{4}$ radians would fall in which quadrant?

(**A**) I (**B**) II (**C**) III (**D**) IV (**E**) On the x or y axis

5. Let $ABCDEF$ be a regular hexagon, with a side length of 5. What is the area of quadrilateral $ABDE$?

(**A**) 25 (**B**) $25\sqrt{2}$ (**C**) $25\sqrt{3}$ (**D**) 50 (**E**) None of these

6. Which of the following pairs (a, x) is a solution to the equation

$$\frac{4a + x}{a + x} - 1 = \frac{12a}{3a + x}?$$

(**A**) $(\sqrt{5}, 3\sqrt{5})$ (**B**) $(3\sqrt{5}, \sqrt{5})$ (**C**) $(-\sqrt{5}, 3\sqrt{5})$ (**D**) $(-3\sqrt{5}, \sqrt{5})$ (**E**) None of these

7. Find the positive real solution to the equation $2x^2 - 3x^{6/5} - 9x^{2/5} = 0$.

(**A**) $3\sqrt[4]{3}$ (**B**) $\frac{3\sqrt[4]{3}}{4}$ (**C**) $\sqrt[5]{81}$ (**D**) $\frac{\sqrt[5]{162}}{2}$ (**E**) None of these

8. Find the sum of the reciprocals of the two solutions to the equation $(2x - 3)(x - 4) = 33$.

(**A**) $\frac{55}{666}$ (**B**) $\frac{39}{74}$ (**C**) $\frac{2}{5}$ (**D**) $-\frac{11}{21}$ (**E**) None of these

9. The volume of a sphere of radius 8 cm is equal to the volume of a right square based pyramid, with a base side length of 32. What is the height of the pyramid?

(**A**) $\frac{\pi}{4}$ (**B**) $\frac{3\pi}{4}$ (**C**) π (**D**) 2π (**E**) None of these

10. A point (x, y) is 5 units from the origin on the line $y = -2x$. Find x^2y^2.

(**A**) 100 (**B**) 400 (**C**) 500 (**D**) 2500 (**E**) None of these

11. Find the mean of all solutions to the equation $2\sin x - 2\sqrt{3}\cos x - \sqrt{3}\tan x + 3 = 0$ on the interval $[0, 2\pi)$.

(**A**) $\frac{5}{6}\pi$ (**B**) $\frac{13}{6}\pi$ (**C**) $\frac{7}{12}\pi$ (**D**) $\frac{11}{12}\pi$ (**E**) None of these

12. If (x, y) is a solution to the system of inequalities below, find the minimum value of y.

$$\begin{cases} 3x + 2y \geq 4 \\ 5x - 6y \geq 16 \\ 2x - y \leq 12 \end{cases}$$

(**A**) -4 (**B**) $-\frac{3}{2}$ (**C**) 0 (**D**) 2 (**E**) None of these

13. Let $f(x) = |x - 4|$ and $g(x) = \sqrt{(x-2)^2} - 1$. When the solutions of the equation $(g \circ f)(x) = 0$ are written in increasing order, they form an arithmetic progression. What would the next number in that progression be?

(**A**) 5 (**B**) 9 (**C**) 13 (**D**) 17 (**E**) None of these

14. Solve for x in the equation $\log_4(81)\log_3(x) = \log_2(625)$.

(**A**) 27 (**B**) 36 (**C**) 75 (**D**) 125 (**E**) None of these

15. A convex polygon has interior angles that measure, in degrees, $2x$, $6x + 5$, $8x - 5$, $7x$ and $5x - 20$. Find the difference in the measure of the largest angle of this polygon and its smallest.

(**A**) 40° (**B**) 55° (**C**) 105° (**D**) 115° (**E**) None of these

16. Find sum of the squares of all solutions to the equation $x^4 - 4x^3 - 9x^2 + 12x + 18 = 0$.

(**A**) 14 (**B**) 34 (**C**) 58 (**D**) 94 (**E**) None of these

17. Find the sum of solutions to the equation $\log_2(x)\,(\log_2(x) - 8) = -15$.

(**A**) 8 (**B**) 16 (**C**) 34 (**D**) 40 (**E**) None of these

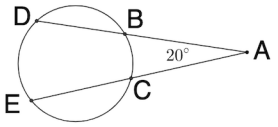
18. In right triangle ABE, $\angle ABE$ is a right angle with $AB = 14$. Let D be a point on $\overline{BE}$ and C a point on $\overline{AE}$ such that $\overline{CD} \perp \overline{BE}$, $BD = 12$, and $DE = 16$. What is the value of $\sin\angle DBC$?

(**A**) $\frac{\sqrt{13}}{13}$ (**B**) $\frac{2\sqrt{13}}{13}$ (**C**) $\frac{3\sqrt{13}}{13}$ (**D**) $\frac{4\sqrt{13}}{13}$ (**E**) None of these

19. For how many values of k does the equation $x^3 + 3x^2 + 3x + k = 0$ have a single real solution?

(**A**) 0 (**B**) 1 (**C**) 3 (**D**) infinitely many (**E**) None of these

20. In the figure shown, points D, B, E, and C are on the circle, and rays $\overrightarrow{DB}$ and $\overrightarrow{EC}$ meet at point A, with $m\angle BAC = 20°$. If $m\widehat{BD} = m\widehat{DE} = m\widehat{CE}$, find the measure of angle $\angle DBE$.



(**A**) 20° (**B**) 40° (**C**) 50° (**D**) 100° (**E**) None of these

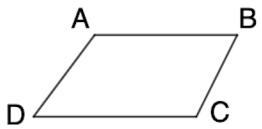
21. Which of the following is a square root of $4a^2 + 4ax^2 - 4abx + x^4 + b^2x^2 - 2bx^3$?

(**A**) $2a - bx - x^2$ (**B**) $2a + bx + x^2$ (**C**) $2a + bx - x^2$ (**D**) $2a - bx + x^2$ (**E**) None of these

22. If the four complex solutions to $x^4 - 16 = 0$ are the four corners of a square in the complex plane, find the area of the square.

(**A**) 4 (**B**) 8 (**C**) 16 (**D**) 32 (**E**) None of these

23. Let $\square ABCD$ be a quadrilateral with $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$, $m\angle ADC = 45°$, $m\angle BCD = 120°$, $AD = 12\sqrt{2}$ and $CD = 27$. Find the length of $\overline{AB}$.



(**A**) $15 + 4\sqrt{2}$ (**B**) $15 + 12\sqrt{2}$ (**C**) $15 + 2\sqrt{3}$

(**D**) $15 + 4\sqrt{3}$ (**E**) None of these

24. If the intersection of the slant asymptote and the vertical asymptote of the graph of the function $y = \dfrac{2x^2 - x - 3}{x - 2}$ is represented as (a, b), find $a + b$.

(**A**) 9 (**B**) -9 (**C**) -3 (**D**) 3 (**E**) None of these

25. What is the area of the circle defined by the equation $x^2 + 6x + y^2 - 4y - 3 = 0$?

(**A**) 4π (**B**) 16π (**C**) 64π (**D**) 256π (**E**) None of these

26. Define $f(x) = 2x^2 + 6x - 7$ on the interval $[-1.5, \infty)$. What is the value of $f^{-1}(13)$?

(**A**) 5 (**B**) 2 (**C**) $\dfrac{1}{409}$ (**D**) $\dfrac{1}{2023}$ (**E**) None of these

27. Let O_1 and O_2 be concentric circles with the radius of O_2 larger than the radius of O_1. The length of a cord of O_2, which is tangent to O_1 is 22 units. Find the area of the region inside O_2 but outside O_1.

(**A**) 25π (**B**) 104π (**C**) 121π (**D**) 220π (**E**) None of these

28. Two fair six-sided dice are rolled. Given that at least one of the die showed an even number, what is the probability that the product of the two numbers shown by the dice is divisible by 4?

(**A**) $\dfrac{2}{3}$ (**B**) $\dfrac{3}{4}$ (**C**) $\dfrac{5}{9}$ (**D**) $\dfrac{5}{12}$ (**E**) None of these

29. In the equation $x^2 + 3hx = 1$ the sum of the squares of the solutions is 3. Find $|h|$.

(**A**) $\dfrac{2}{9}$ (**B**) $\dfrac{1}{3}$ (**C**) $\dfrac{\sqrt{2}}{3}$ (**D**) $\dfrac{\sqrt{6}}{3}$ (**E**) None of these

30. Use the fact that $\log_{10} 2 \approx 0.301$ to find the approximate value of $\log_{10} 50$.

(**A**) 0.7525 (**B**) 1.301 (**C**) 1.505 (**D**) 1.699 (**E**) None of these

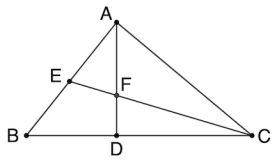
31. If $\tan\theta = -\dfrac{12}{5}$, and $\sin\theta > 0$, find $\cos\theta$.

(**A**) $-\dfrac{5}{13}$ (**B**) -5 (**C**) 5 (**D**) $\dfrac{5}{13}$ (**E**) None of these

32. Recall that the value $x \mod n$, for positive integers x and n, is the remainder after x is divided by n. Calculate the value of $11^{111} \mod 1000$.

(**A**) 111 (**B**) 211 (**C**) 311 (**D**) 411 (**E**) None of these

33. In triangle $\triangle ABC$, $\angle BAC$ is a right angle, $BC = 8$, and $AC = 4\sqrt{3}$. Let $\overline{CE}$ be the median to side $\overline{AB}$ and $\overline{AD}$ an altitude to side $\overline{BC}$. Segments $\overline{CE}$ and $\overline{AD}$ intersect at point F. What is the length of $\overline{AF}$?



(**A**) $\sqrt{3}$ (**B**) $\frac{4\sqrt{3}}{3}$ (**C**) $\frac{6\sqrt{3}}{5}$ (**D**) $\frac{8\sqrt{3}}{7}$ (**E**) None of these

34. Let $f(x) = x^5 - 28x^3 + 75x$. The function $f(x)$ has two non-zero rational roots, one of which is $x = 5$ and the other we will call a. Find $a^2 + 3a + 2$.

(**A**) -38 (**B**) 0 (**C**) 6 (**D**) 12 (**E**) None of these

35. Solve for x if $2^{3x+2} + 8^x = \frac{5}{16}$.

(**A**) $-\frac{4}{3}$ (**B**) $-\frac{3}{4}$ (**C**) -2 (**D**) $-\frac{1}{2}$ (**E**) None of these

36. For $f(x)$ defined below, find the sum of all values for which $f(x) = 1$.

$$f(x) = \begin{cases} x^2 + 6x + 6 & x \leq 0 \\ 2x - 5 & 0 < x < 3 \\ \sqrt{x-3} & x \geq 3 \end{cases}.$$

(**A**) -2 (**B**) -1 (**C**) 1 (**D**) 3 (**E**) None of these

37. One bag contains 5 white and 3 red balls, and a second bag contains 4 white and 5 red balls. From one of them, chosen at random, two balls are drawn without replacement. Find the probability that the balls are different colors.

(**A**) $\frac{9}{34}$ (**B**) $\frac{37}{72}$ (**C**) $\frac{72}{289}$ (**D**) $\frac{275}{504}$ (**E**) None of these

38. What is the slope of the line that bisects the acute angle given by the lines $y = 0$ and $y = x$?

(**A**) $\frac{1}{2}$ (**B**) $\sqrt{2}$ (**C**) $\sqrt{2} - 1$ (**D**) $\frac{\sqrt{2}}{2}$ (**E**) None of these

39. Let $p(x)$ be a polynomial that has $x - 1$, $x^2 - 1$, $x^3 - 1$ and $x^4 - 1$ as factors. What is the smallest possible degree of $p(x)$?

(**A**) 6 (**B**) 7 (**C**) 8 (**D**) 10 (**E**) None of these

40. How many integers are in the solution set to $|x^2 - 2x - 2| > |x^2 - 2x + 2|$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 4 (**E**) None of these

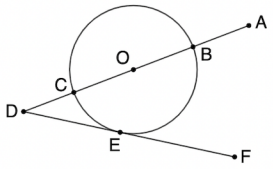
41. The point $(2, -3)$ goes to the point $(4, -5)$ after a dilation centered at point (a, b) with scale factor $k = \frac{5}{2}$. What is $a + b$?

(**A**) $\frac{2}{3}$ (**B**) $-\frac{2}{3}$ (**C**) 1 (**D**) -1 (**E**) None of these

42. Which of the following polynomials has a graph which does not contain points in the second quadrant?

(**A**) x^2+7x-4 (**B**) $-x^2+6$ (**C**) x^4-3x^2+x (**D**) $-x^4+2x^3-1$ (**E**) $-4x^5+2x-7$

43. In the figure, the circle is centered at O, $\overline{AD}$ is a secant passing through O and intersecting the circle at points B and C, $\overline{DF}$ is tangent to the circle at point E, and $\overline{BF}$ is perpendicular to $\overline{AD}$. If $m\widehat{BE}=120°$, and the diameter of the circle is 10, find the length of $\overline{EF}$.



(**A**) $5\sqrt{3}$ (**B**) 10 (**C**) $5\sqrt{3}+5$ (**D**) 15 (**E**) None of these

44. Find the product of all solutions to the equation $\cos x \csc x - 2 = 2\cos x - \csc x$ that lie in the interval $[0, 2\pi)$.

(**A**) $\frac{2\pi^2}{9}$ (**B**) $\frac{2\pi^3}{9}$ (**C**) $\frac{5\pi^2}{36}$ (**D**) $\frac{5\pi^3}{36}$ (**E**) None of these

45. How many numbers greater than a million can be formed using the digits 2, 3, 0, 3, 4, 2, 3 each one time?

(**A**) 360 (**B**) 720 (**C**) 2160 (**D**) 4320 (**E**) None of these

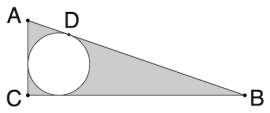
46. Let θ be the acute angle between the lines $y=x$ and $y=2x$. Find $\sin\theta$.

(**A**) $\frac{\sqrt{5}}{5}$ (**B**) $\frac{\sqrt{6}}{6}$ (**C**) $\frac{\sqrt{10}}{10}$ (**D**) $\frac{3\sqrt{10}}{10}$ (**E**) None of these

47. What is the coefficient of the x term of the least degree polynomial that passes through the points $(1,1)$, $(0,-1)$, and $(2,4)$?

(**A**) -3 (**B**) $-\frac{5}{2}$ (**C**) $\frac{1}{2}$ (**D**) $\frac{3}{2}$ (**E**) None of these

48. Let $\triangle ABC$ be right triangle with right angle at vertex C, and let a circle be inscribed in $\triangle ABC$ with a point of tangency to side $\overline{AB}$ at D. If $AD=7$ units, and the radius of the circle is 5 units, find the area of the shaded region, in units squared, rounded to the nearest integer.



(**A**) 131 (**B**) 135 (**C**) 139 (**D**) 143 (**E**) None of these

49. If x^2+2x+5 is a factor of x^4+px^2+q, then what is $p+q$?

(**A**) 10 (**B**) 19 (**C**) 26 (**D**) 31 (**E**) None of these

50. Trent is preparing a salad. He can make his salad with just lettuce, or he can make it with lettuce and any of the following toppings: cucumbers, tomatoes, onions, carrots, mushrooms, croutons, and/or cheese. How many variations of the salad are possible?

(**A**) 49 (**B**) 64 (**C**) 128 (**D**) 256 (**E**) None of these