The 41st Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

Written Round: February 25-27, 2022 at your school

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COMPREHENSIVE EXAMINATION

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. The quartic polynomial $3x^4 - 8x^3 - 50x^2 - 57x - 18$ can be factored as $(x^2 - 5x - 6)(ax^2 + bx + c)$. Find $a + b + c$.

(**A**) -23 (**B**) 7 (**C**) $\boxed{13}$ (**D**) 27 (**E**) None of these

2. Two roots of the polynomial $3x^3 + ax^2 - 5x + 10$ are r and $-r$ for r a real number. What is the value of a?

(**A**) $\boxed{-6}$ (**B**) 6 (**C**) -4 (**D**) 4 (**E**) None of these

3. Two sides of a triangle measure 8 inches and 12 inches. If the altitude to the 8 inch side is 6 inches, what is the altitude to the 12 inch side?

(**A**) $\boxed{4}$ (**B**) 6 (**C**) 8 (**D**) 10 (**E**) None of these

4. Find the value of $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots + 19^2 - 20^2$.

(**A**) -13175 (**B**) $\boxed{-210}$ (**C**) 1330 (**D**) 2870 (**E**) None of these

5. An isosceles triangle has two sides of length 10 and one of length 8. What is the area of the triangle?

(**A**) 40 (**B**) 80 (**C**) $4\sqrt{21}$ (**D**) $\boxed{8\sqrt{21}}$ (**E**) None of these

6. A history final exam has 25 questions. If the questions were all equally weighted, Norman would have earned a 72%. However, the first twenty questions are worth three points each, and the final five questions are worth eight points each. If Norman got a 64% on the test with this point distribution, how many eight point questions did Norman get correct?

(**A**) 1 (**B**) $\boxed{2}$ (**C**) 3 (**D**) 4 (**E**) None of these

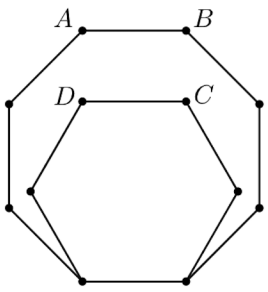
7. The graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ passes through the points $(6, 1598)$, $(-3, -292)$, $(0, -4)$ and $(4, 464)$, with a, b, c, and d all integer coefficients. What is the value of the product $abcd$?

(**A**) -945 (**B**) -362 (**C**) 525 (**D**) 810 (**E**) $\boxed{\text{None of these}}$

8. If a and b are the solutions to the equation $(2x + 3)(3x - 1) = 7$, find $|a| + |b|$.

(**A**) $\dfrac{8}{3}$ (**B**) $\dfrac{14}{3}$ (**C**) $-\dfrac{7}{6}$ (**D**) $\boxed{\dfrac{17}{6}}$ (**E**) None of these

9. A regular hexagon of side length 4 is embedded in a regular octagon of side length 4, as shown. What is the area of quadrilateral $ABCD$?



(**A**) $16\sqrt{2}$ (**B**) $16\sqrt{3}$ (**C**) $\boxed{16 + 16\sqrt{2} - 16\sqrt{3}}$

(**D**) $16 + 16\sqrt{3} - 16\sqrt{2}$ (**E**) None of these

10. The graph of points satisfying the equation $y^2 = (x - 4)^2 + 3$ gives a(n)

(**A**) Circle (**B**) Line (**C**) Parabola (**D**) Ellipse (**E**) $\boxed{\text{Hyperbola}}$

11. Find the sum of all solutions to the equation $\cos(3x) + \cos(2x) + \cos(x) = 0$ on the interval $[0, 2\pi)$.

(**A**) 2π (**B**) 3π (**C**) 5π (**D**) $\boxed{6\pi}$ (**E**) None of these

12. Let a and b be real numbers, with $a \neq 0$. The quadratic equation $a^2x^2 - 4abx - 5b^2 = 0$ has a unique real solution when:

(**A**) $a = 1$ (**B**) $\boxed{b = 0}$ (**C**) $a + b = 0$ (**D**) $\dfrac{5b^2}{a^2} = 1$ (**E**) None of these

13. Find the distance in the complex plane between the two solutions to the equation $x^2 - 8i = 0$.

(**A**) 4 (**B**) $\boxed{4\sqrt{2}}$ (**C**) 8 (**D**) $8\sqrt{2}$ (**E**) None of these

14. The equation $x^3 + 27 = 0$ has one real solution r and two complex solutions of the form $a + bi$ and $c + di$. Find $a^2 + b^2 + c^2 + d^2$.

(**A**) $\dfrac{9}{2}$ (**B**) 9 (**C**) $\boxed{18}$ (**D**) 27 (**E**) None of these

15. Four distinct points A, B, C, and D lie clockwise in that order on a circle. Chords $\overline{AB}$ and $\overline{DC}$ can be extended at B and C, respectively, to meet at point P, which is external to the circle. If $AB = 10$, $CD = 7$, $BP = 8$ and $m\angle APD = 60°$, find the area of the circle.

(**A**) $\dfrac{243}{4}\pi$ (**B**) $\dfrac{97}{2}\pi$ (**C**) $\boxed{73\pi}$ (**D**) 81π (**E**) None of these

16. Which of the following is NOT a valid trigonometric identity?

(**A**) $\boxed{1 + \tan^2\theta = \cot^2\theta}$ (**B**) $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

(**C**) $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$ (**D**) $\csc\theta = \sec(\frac{\pi}{2} - \theta)$ (**E**) None of these

17. Define values m and n in terms of variables a and x by $m = 2a^3 - 2ax^2$ and $n = 2a^3 + 4a^2x + 2ax^2$. When $a = 40$ and $x = 35$, what is the greatest common divisor of m and n?

(**A**) 5 (**B**) 2,000 (**C**) 6,000 (**D**) $\boxed{30{,}000}$ (**E**) None of these

18. A metal strip is shaped into a right triangle in such a way that it has its three sides in arithmetic progression. If the length of the shorter leg is 9, find the length of the hypotenuse.

(**A**) 11 (**B**) 13 (**C**) $\boxed{15}$ (**D**) 17 (**E**) None of these

19. The system of equations below has a unique solution (a, b). Find the value of $3a + 2b$.

$$\begin{cases} 4^x 8^y = 32 \\ x - y = 10 \end{cases}$$

(**A**) 5 (**B**) $\dfrac{15}{2}$ (**C**) $\boxed{15}$ (**D**) 25 (**E**) None of these

20. The area of a rectangle is 10 and the length of its diagonal is 5. Find the difference of the square of the longer side and the square of the shorter side.

(**A**) 7 (**B**) $\boxed{15}$ (**C**) 21 (**D**) 28 (**E**) None of these

21. Find the product of the largest and smallest solutions of the equation $(2x^2+4x-3)^2 = 9-4x-2x^2$.

(**A**) -6 (**B**) -3 (**C**) $\boxed{-\frac{5}{2}}$ (**D**) 0 (**E**) None of these

22. In triangle ABC, the interior angle at vertex B measures $72°$, and the exterior angle at vertex A measures $145°$. What is the measure of the interior angle at vertex C, in degrees?

(**A**) 37 (**B**) 45 (**C**) 54 (**D**) $\boxed{73}$ (**E**) None of these

23. For what values of $x > 1$ does the inequality $x^2 \leq (\ln x)^{\ln x}$ hold?

(**A**) $x \geq \sqrt{e}$ (**B**) $x \geq e$ (**C**) $x \geq e^{2e}$ (**D**) $\boxed{x \geq e^{(e^2)}}$ (**E**) None of these

24. A jar contains 10 Red, 8 Purple, and 7 Blue marbles. If you draw two marbles without replacement, what is the probability you get one red and one blue?

(**A**) $\frac{14}{125}$ (**B**) $\frac{83}{120}$ (**C**) $\frac{7}{60}$ (**D**) $\boxed{\frac{7}{30}}$ (**E**) None of these

25. For $f(x)$ defined as the piecewise function below, find the sum of all real values of x for which $f(x) = 0$.

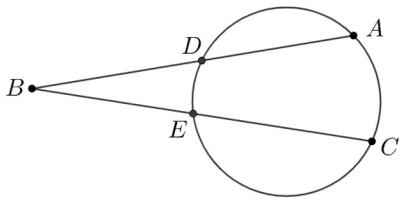
$$f(x) = \begin{cases} |x+3| & \text{for } x \leq -1 \\ x^2+3x-4 & \text{for } -1 < x < 2 \\ \log_5(\frac{1}{5}x) & \text{for } x \geq 2 \end{cases}$$

(**A**) -6 (**B**) -2 (**C**) -1 (**D**) $\boxed{3}$ (**E**) None of these

26. In the figure shown, the measure of angle $\angle DBE$ is $38°$, and the measure of the minor arc $\widehat{DE}$ is $40°$. What is the measure of minor arc $\widehat{AC}$?

(**A**) $36°$ (**B**) $39°$ (**C**) $78°$

(**D**) $\boxed{116°}$ (**E**) None of these



27. Find the smallest solution to the equation $\log_2(x) + \log_{\frac{1}{2}}\left(\frac{1}{x+1}\right) = \log_8(27)$.

(**A**) $\boxed{\frac{-1+\sqrt{13}}{2}}$ (**B**) $\frac{-1-\sqrt{13}}{2}$ (**C**) $\frac{-1+\sqrt{37}}{2}$ (**D**) $\frac{-1-\sqrt{37}}{2}$ (**E**) None of these

28. What is the largest value of y for which a pair (x, y) satisfies $4x^2 + 8y^2 = 4x + 16y + 9$?

(**A**) $\frac{1}{2}$ (**B**) $\frac{5}{4}$ (**C**) $\boxed{\frac{5}{2}}$ (**D**) $\frac{13}{4}$ (**E**) None of these

29. If $a + b = 4$, and $a^2 + b^2 = 12$, then what is $a^4 + b^4$?

(**A**) 112 (**B**) $\boxed{136}$ (**C**) 144 (**D**) 256 (**E**) None of these

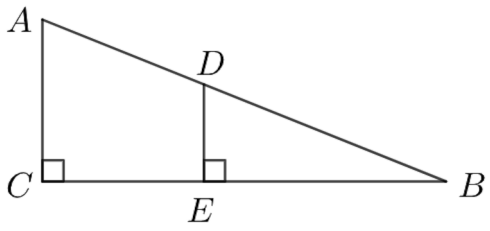
30. Find the radius in units of a sphere for which its volume in cubic units is equal to twice its surface area in square units.

(**A**) 2 (**B**) 3 (**C**) 4 (**D**) $\boxed{6}$ (**E**) None of these

31. If $f(x) = \dfrac{x^{26} + x^{24} + 2x^{22}}{x - 1}$, find $f(i)$, where i is the imaginary unit.

(**A**) $-1 - i$ (**B**) $-1 + i$ (**C**) $1 - i$ (**D**) $\boxed{1 + i}$ (**E**) None of these

32. In triangle ABC given, points D and E lie on sides $\overline{AB}$ and $\overline{BC}$, respectively, with $AC = 3$, $CE = 3$ and $BE = 6$. Given that $\angle ACB$ and $\angle DEB$ are both right angles, find $\cos(\angle ADE)$.



(**A**) $-\dfrac{3\sqrt{10}}{10}$ (**B**) $\boxed{-\dfrac{\sqrt{10}}{10}}$ (**C**) $\dfrac{\sqrt{10}}{10}$

(**D**) $\dfrac{3\sqrt{10}}{10}$ (**E**) None of these

33. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, for $-1 < x < 1$, write $f\left(\dfrac{3x + x^3}{1 + 3x^2}\right)$ in terms of $f(x)$.

(**A**) $f(x^3)$ (**B**) $[f(x)]^3$ (**C**) $f(x+3)$ (**D**) $f(3x)$ (**E**) $\boxed{3f(x)}$

34. A harshad number is an integer number which is divisible by the sum of its digits. Which of the following is NOT a harshad number?

(**A**) 2022 (**B**) 2023 (**C**) 2024 (**D**) 2025 (**E**) $\boxed{2026}$

35. In triangle ABC, let D be the midpoint of side $\overline{AB}$ and E be the midpoint of side $\overline{AC}$. If $BC = 9x+2$ and $DE = 5x - 1$, find the length of $\overline{BC}$.

(**A**) 20 (**B**) 29 (**C**) $\boxed{38}$ (**D**) 47 (**E**) None of these

36. What is the shortest distance from the point $(6, 5)$ to the line $2x + 3y = 1$?

(**A**) $\sqrt{13}$ (**B**) $\boxed{2\sqrt{13}}$ (**C**) $4\sqrt{13}$ (**D**) $6\sqrt{13}$ (**E**) None of these

37. Find the height of a square pyramid formed using four equilateral triangles whose sides all have length 2.

(**A**) $\boxed{\sqrt{2}}$ (**B**) $\sqrt{3}$ (**C**) $\sqrt{5}$ (**D**) $\sqrt{6}$ (**E**) None of these

38. Find the number of real solutions to the equation $\sqrt[2]{x+1} = \sqrt[3]{3x-1}$.

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

39. Given that the graph of the parabola $y = ax^2 - 32x + 4$ has an axis of symmetry of $x = 8$, find the value of a.

(**A**) $\dfrac{63}{64}$ (**B**) $\dfrac{3}{2}$ (**C**) $\boxed{2}$ (**D**) 4 (**E**) None of these

40. If $\sin\theta + \cos\theta = \frac{1}{2}$, what is the value of $\sin^2\theta\cos^2\theta$?

(**A**) 1 (**B**) $\frac{1}{144}$ (**C**) $\frac{1}{64}$ (**D**) $\boxed{\frac{9}{64}}$ (**E**) None of these

41. A line through the points $(m, -9)$ and $(7, m)$ has slope m. Find the value of m.

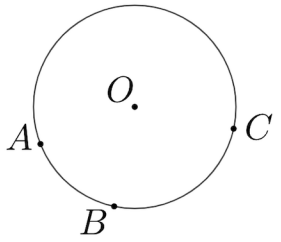
(**A**) -3 (**B**) -2 (**C**) 2 (**D**) $\boxed{3}$ (**E**) None of these

42. The roots of the equation $x - 10 = -\frac{24}{x}$ represent two sides of a scalene triangle. Which of the following would be a valid third side to make the triangle an acute triangle?

(**A**) 1 (**B**) 3 (**C**) 9 (**D**) 11 (**E**) $\boxed{\text{None of these}}$

43. In the figure shown, point O is the center of the circle and A, B and C are three points on the circle. Suppose that $OA = AB = 2$, and angle $\angle OAC$ measures $10°$. Find the measure of minor arc $\overset{\frown}{BC}$ in degrees.

(**A**) 50 (**B**) 70 (**C**) 80 (**D**) $\boxed{100}$ (**E**) None of these



44. If $f(x)$ is an odd function, which of the following is NOT an even function?

(**A**) $f(|x|)$ (**B**) $|f(x)|$ (**C**) $f(x^2)$ (**D**) $[f(x)]^2$ (**E**) $\boxed{f(-x)}$

45. Find the remainder when $f(x) = (x+4)(x^2-7) + (x+2)(2x-4) - (x+1)(x^3+2)(2x-1)$ is divided by $x - 1$.

(**A**) $\boxed{-42}$ (**B**) -34 (**C**) -32 (**D**) -24 (**E**) None of these

46. Find the smallest solution of the equation $2^{x-1} + 2^{-(x+1)} = 4$.

(**A**) $\log_2(4 - \sqrt{17})$ (**B**) $\boxed{\log_2(4 - \sqrt{15})}$

(**C**) $\log_2(4 - \sqrt{14})$ (**D**) $\log_2(4 - \sqrt{13})$ (**E**) None of these

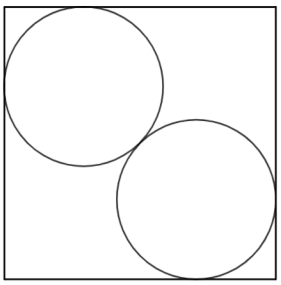
47. Find the sum of values of x for which the graph of the function $y = \dfrac{4}{2\sin^2(x) + \sin(x) - 1}$ has a vertical asymptote on the interval $[0, 2\pi)$.

(**A**) $\boxed{\frac{5\pi}{2}}$ (**B**) $\frac{5\pi}{3}$ (**C**) 2π (**D**) 3π (**E**) None of these

48. The function $f(x) = x^4 + 4x^3 + 10x^2 + 12x + 12$ has a minimum value of $y = b$ occurring at $x = a$. Find $a + b$.

(**A**) 3 (**B**) $\boxed{6}$ (**C**) 10 (**D**) 12 (**E**) None of these

49. Two identical circles are placed into a square in such a way that they are tangent to each other at a single point, and each circle is tangent to the square at two points, as shown. If the radius of each circle is 1, what is the area of the square?



(**A**) $\frac{25}{2}$ (**B**) $\frac{49}{4}$ (**C**) $3+2\sqrt{2}$ (**D**) $\boxed{6+4\sqrt{2}}$ (**E**) None of these

50. The equation $2x+3y-4z=5$ has infinitely many integer solutions. Which of the following cannot be a value of y for any integer values of x and z?

(**A**) $\boxed{2}$ (**B**) 3 (**C**) 5 (**D**) 9 (**E**) None of these