The 40th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

Written Round: April 9 – 11, 2021 at your school

COMPREHENSIVE EXAMINATION

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. What value of b is needed so that the expression $3x^3 + bx^2 + 17x + 7$ has a factor of $x^2 - 4x + 7$?

(**A**) -13 (**B**) $\boxed{-11}$ (**C**) 11 (**D**) 13 (**E**) None of these

2. Find the coefficient on the x^5 term in the expansion of $(x + i)^{10}$, where i is the imaginary unit.

(**A**) i (**B**) $2i$ (**C**) $\boxed{252i}$ (**D**) $30240i$ (**E**) None of these

3. Find the center of the ellipse given by $8x^2 + 9y^2 + 32x + 54y = -41$.

(**A**) $(2, 3)$ (**B**) $(-4, -6)$ (**C**) $(-16, -27)$ (**D**) $(16, 27)$ (**E**) $\boxed{\text{None of these}}$

4. A Gaussian integer is defined to be an ordered pair (x, y) where x and y are both integers. A Gaussian integer (x, y) is said to be *prime* if x^2+y^2 is prime. Which of the following is a prime Gaussian integer?

(**A**) $(1, 3)$ (**B**) $(3, 5)$ (**C**) $(4, 4)$ (**D**) $\boxed{(4, 5)}$ (**E**) None of these

5. Find the minimum value of the function $y = 2\sin x - \cos 2x + 4$.

(**A**) 1 (**B**) $\boxed{\frac{5}{2}}$ (**C**) 3 (**D**) $\frac{7}{2}$ (**E**) None of these

6. The following system of equations has one solution in Quadrant IV. Find the distance from that solution to the origin.

$$\begin{cases} x^2 + y^2 + xy + 2x + 2y = 5 \\ 5x^2 + 5y^2 + 2xy + 8x + 8y = 24 \end{cases}$$

(**A**) $\boxed{\frac{\sqrt{26}}{3}}$ (**B**) $\frac{\sqrt{34}}{3}$ (**C**) $\frac{5\sqrt{2}}{3}$ (**D**) $\frac{\sqrt{82}}{3}$ (**E**) None of these

7. In right triangle ABC, $m\angle C = 30°$, $m\angle B = 90°$, and D is a point on $\overline{BC}$ such that $\overrightarrow{AD}$ is the angle bisector of $\angle BAC$. If $AB = 6$, find CD.

(**A**) $\boxed{4\sqrt{3}}$ (**B**) $6\sqrt{3}$ (**C**) 10 (**D**) 12 (**E**) None of these

8. Find the sum of the squares of all x for which $(x + 1)(x - 2) = 10$.

(**A**) 5 (**B**) 10 (**C**) $\boxed{25}$ (**D**) 225 (**E**) None of these

9. The circle $x^2 + y^2 - 8x + ky = -11$ has its center on the line $y = x + 1$. Find the radius of the circle.

(**A**) $\sqrt{14}$ (**B**) $\boxed{\sqrt{30}}$ (**C**) $\sqrt{55}$ (**D**) $\sqrt{134}$ (**E**) None of these

10. What is the minimum value of $x^2 + y^2 - 2(xy - x + y) + 5$, for $x, y \geq 0$?

(**A**) -10 (**B**) $\boxed{4}$ (**C**) 5 (**D**) There is no minimum value (**E**) None of these

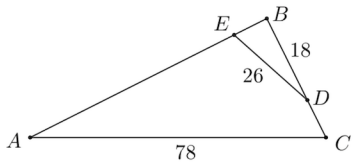
11. If the diagonals of a rhombus are 14 and 22 units long, find the area of the rhombus.

(**A**) $36\ u^2$ (**B**) $77\ u^2$ (**C**) $\boxed{154\ u^2}$ (**D**) $308\ u^2$ (**E**) None of these

12. Find the sum of solutions to the equation $(2^x-4)^3+(4^x-2)^3=(4^x+2^x-6)^3$.

(**A**) 1 (**B**) 2 (**C**) $\frac{5}{2}$ (**D**) $\boxed{\frac{7}{2}}$ (**E**) None of these

13. In triangle $\triangle ABC$ shown, point E is on $\overline{AB}$, point D is on $\overline{BC}$, $AC=78$, $BD=18$, and $DE=26$. If $CD=BE$ and $\angle BDE\cong\angle BAC$, find the perimeter of $\triangle ABC$.



(**A**) $\boxed{159}$ (**B**) 183 (**C**) 224 (**D**) 240 (**E**) None of these

14. Which of the following is NOT a root of the polynomial $f(x)=6x^5-19x^4-79x^3+324x^2-220x-112$?

(**A**) $\frac{7}{2}$ (**B**) -4 (**C**) $-\frac{1}{3}$ (**D**) 2 (**E**) $\boxed{\frac{2}{5}}$

15. If a and b are the solutions to the equation $x^2+4x+1=0$, find $\frac{a}{b}+\frac{b}{a}$.

(**A**) -10 (**B**) -4 (**C**) 8 (**D**) $\boxed{14}$ (**E**) None of these

16. In triangle $\triangle ABC$, the midpoint of side $\overline{AB}$ is $(\frac{1}{2},3)$, the midpoint of $\overline{AC}$ is $(3,-3)$, and the midpoint of side $\overline{BC}$ is $(\frac{9}{2},-1)$. Find the perimeter of $\triangle ABC$.

(**A**) $9+2\sqrt{2}$ (**B**) $9+4\sqrt{2}$ (**C**) $18+4\sqrt{2}$ (**D**) $\boxed{18+8\sqrt{2}}$ (**E**) None of these

17. Find the exact value of $\cos\left(\sin^{-1}\left(\frac{7}{12}\right)\right)$.

(**A**) $\boxed{\frac{\sqrt{95}}{12}}$ (**B**) $\frac{\sqrt{193}}{12}$ (**C**) $\frac{12}{\sqrt{95}}$ (**D**) $\frac{12}{\sqrt{193}}$ (**E**) None of these

18. A non-vertical line L intersects the graph of $f(x)=x^2-6x+3$ at only the point $(4,-5)$. Find the y-intercept of the line L.

(**A**) -51 (**B**) -21 (**C**) $\boxed{-13}$ (**D**) 51 (**E**) None of these

19. When the point $(3,5)$ is reflected about the line $y=4x-2$, the resulting point is (a,b). Find $a+b$.

(**A**) $\frac{75}{11}$ (**B**) $\boxed{\frac{106}{17}}$ (**C**) $\frac{124}{23}$ (**D**) 8 (**E**) None of these

20. Let A be the measure of an acute angle whose complement measures one-fourth of its supplement. Let B equal the geometric mean of 4 and 36. Find the arithmetic mean of A and B.

(**A**) 24 (**B**) $\boxed{36}$ (**C**) 42 (**D**) 60 (**E**) None of these

21. Let a be the solution to the equation below. Find $f(a)$, where $f(x) = 8x^2 + 2x + 3$.

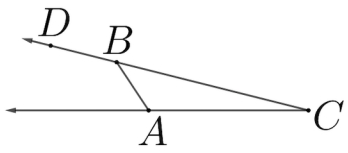
$$\sqrt{1 + \left(\frac{1}{2\sqrt{x}} - \frac{1}{2}\sqrt{x}\right)^2} = 4x + 4x^2.$$

(**A**) $\boxed{4}$ (**B**) $\frac{29}{9}$ (**C**) 9 (**D**) $\frac{44}{9}$ (**E**) None of these

22. Consider the function $f(x) = \frac{2^x + 7}{4^{x-2}} + 9$. What is the value of $f(218902)$, rounded to the nearest integer?

(**A**) 7 (**B**) 8 (**C**) $\boxed{9}$ (**D**) 10 (**E**) None of these

23. Let $\triangle ABC$ be a triangle such that $AB = 4$, $AC = 10$ and $BC = 12$. Let T be a point on $\overrightarrow{CA}$ such that $\overrightarrow{BT}$ bisects the exterior angle $\angle ABD$. Find AT.



(**A**) 2 (**B**) 3 (**C**) 4 (**D**) $\boxed{5}$ (**E**) None of these

24. Amy is sewing non-medical masks for visitors for the local hospital. The fabric for the masks is only sold by the yard, and each yard of fabric can make 8 masks. Each mask also uses 15 inches of elastic, and elastic is also only sold by the yard. What is the smallest number of masks that Amy can make without have any left over supplies? Recall that one yard is equivalent to 36 inches.

(**A**) 8 masks (**B**) $\boxed{24 \text{ masks}}$ (**C**) 48 masks (**D**) 72 masks (**E**) None of these

25. For which of the following (x, y) pairs is the number represented by $12x^4+47x^3y+37x^2y^2-30xy^3-26y^4$ divisible by the number represented by $4x + 9y$?

(**A**) $(11, 12)$ (**B**) $(16, 13)$ (**C**) $\boxed{(19, 20)}$ (**D**) $(22, 18)$ (**E**) None of these

26. If you write $\sqrt{i^3}$ in $a + bi$ form, where i is the imaginary unit, then ab is

(**A**) $\boxed{-\frac{1}{2}}$ (**B**) 0 (**C**) $\frac{1}{4}$ (**D**) $\frac{1}{2}$ (**E**) None of these

27. Find the smallest real number b such that the point (a, b) lies on the graph of $x^2+y^2-\frac{3}{4}x+2y-\frac{1}{8} = 0$.

(**A**) $-\frac{145}{64}$ (**B**) $\boxed{-\frac{17}{8}}$ (**C**) $-\frac{17}{64}$ (**D**) $-\frac{1}{8}$ (**E**) None of these

28. Find the value of $f(f(f(f(1))))$, where f is defined as $f(x) = \begin{cases} x^{-1} + 4, & \text{for } x < -1 \\ x^2 + 1, & \text{for } -1 \le x \le 1 \\ x - 6, & \text{for } x > 1 \end{cases}$.

(**A**) $\boxed{-\frac{9}{4}}$ (**B**) 2 (**C**) $\frac{17}{4}$ (**D**) 16 (**E**) None of these

29. The angles of a pentagon are in arithmetic progression with a common difference between each angle of $4°$. Find the measure of the largest angle.

(**A**) $108°$ (**B**) $\boxed{116°}$ (**C**) $120°$ (**D**) $124°$ (**E**) None of these

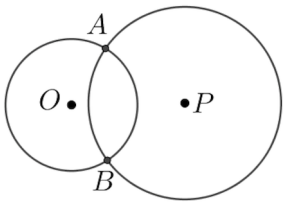
30. A boy is 1 year more than twice as old as his brother. The two boys together are 10 years older than their sister, who is 3 years younger than the older boy. Find the sum of the ages of the three children.

(**A**) 10 (**B**) 27 (**C**) $\boxed{34}$ (**D**) 42 (**E**) None of these

31. Two zeros of the function $f(x) = x^4 - 7x^3 + 9x^2 + 13x - 4$ are $x = 2 - \sqrt{3}$ and $x = 4$. Find the sum of the squares of the remaining two zeros.

(**A**) $\boxed{4\sqrt{3} + 8}$ (**B**) $4\sqrt{3} + 23$ (**C**) 8 (**D**) 23 (**E**) None of these

32. The circle with center O intersects the circle with center P at points A and B, as shown in the figure. If the measure of $\widehat{AB}$ on circle P is $50°$, and the measure of $\widehat{AB}$ on circle O is $80°$, what is the measure of angle $\angle OBP$?



(**A**) $90°$ (**B**) $\boxed{115°}$ (**C**) $130°$ (**D**) $147.5°$ (**E**) None of these

33. Let $f(x) = 3x + 5$. What is the smallest value for which $[f(x)]^2 = f(x^2)$?

(**A**) There is no value of x for which this is true (**B**) This is true for all values of x

(**C**) $\boxed{\dfrac{-15 - \sqrt{105}}{6}}$ (**D**) $\dfrac{12 - \sqrt{127}}{18}$ (**E**) None of these

34. Find the only value of x for which $\left(\dfrac{x^2 - 6x + 9}{x^2 - 7x + 12}\right)\left(\dfrac{x^3 - 4x^2 + 9x - 36}{x^4 - 81}\right) = 15$.

(**A**) $-\dfrac{15}{44}$ (**B**) $\boxed{-\dfrac{44}{15}}$ (**C**) $-\dfrac{2}{15}$ (**D**) $-\dfrac{15}{2}$ (**E**) None of these

35. Find the smallest positive integer a for which the equation $(a + x)^{2/3} + 4(a - x)^{2/3} = 5(a^2 - x^2)^{1/3}$ has only integer solutions for x.

(**A**) 3 (**B**) 5 (**C**) 63 (**D**) $\boxed{65}$ (**E**) None of these

36. If a and b are positive real numbers, and $\ln \sqrt{e} + 125^b i = a + 5i$, where i is the imaginary unit, find $a + b$.

(**A**) $-\dfrac{5}{3}$ (**B**) $-\dfrac{5}{2}$ (**C**) $\dfrac{2}{5}$ (**D**) $\boxed{\dfrac{5}{6}}$ (**E**) None of these

37. The length of a rectangle is reduced by 10 feet, while the width of the same rectangle is increased by 6 feet. If the resulting figure is a square whose area is equal to that of the original rectangle, what is the perimeter of the original rectangle?

(**A**) 32 feet (**B**) $\boxed{68 \text{ feet}}$ (**C**) 128 feet (**D**) 225 feet (**E**) None of these

38. Let x and y represent real numbers. What is the smallest value of y satisfying $x^2 + 6x = y + 3$?

(**A**) -15 (**B**) $\boxed{-12}$ (**C**) -3 (**D**) 6 (**E**) None of these

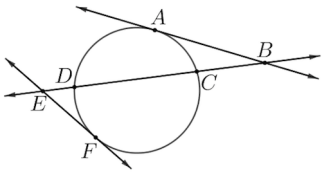
39. A circle is inscribed in a triangle with side lengths 7, 9, and 10. Find the area of the inscribed circle.

(**A**) $\frac{64\pi}{13}$ (**B**) $\boxed{\frac{72\pi}{13}}$ (**C**) $\frac{49\pi}{26}$ (**D**) $\frac{75\pi}{26}$ (**E**) None of these

40. The sum of first two terms of an infinite geometric progression is equal to 5, and each term is 3 times the sum of all the terms that follow it. Find the first term.

(**A**) $\frac{5}{2}$ (**B**) 3 (**C**) $\frac{7}{2}$ (**D**) $\boxed{4}$ (**E**) None of these

41. Line $\overleftrightarrow{AB}$ is tangent to the circle at point A, line $\overleftrightarrow{BE}$ intersects the circle at points C and D, and line $\overleftrightarrow{EF}$ is tangent to the circle at point F. If $AB = 6$, $BC = 4.5$, and $BE = 10$, find EF.



(**A**) 2 (**B**) 6 (**C**) $\sqrt{7}$ (**D**) $\boxed{\sqrt{11}}$ (**E**) None of these

42. Let $f^{-1}(x)$ be the inverse of the function $f(x) = x^3 + 7x + 2$. Find the sum of all zeros of $f^{-1}(x)$.

(**A**) -7 (**B**) 0 (**C**) 3 (**D**) 14 (**E**) $\boxed{\text{None of these}}$

43. If $\sec x - \cos^2 x = \sin^2 x$, what is $\tan x$?

(**A**) -1 (**B**) $\boxed{0}$ (**C**) 1 (**D**) $\sqrt{3}$ (**E**) None of these

44. Suppose n is a positive integer. For how many values of n is $n^2 - 12n + 39$ a perfect square?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) None of these

45. In rectangle $ABCD$, points F and G lie on $\overline{AB}$ such that $AF = FG = GB$, and E is the midpoint of $\overline{CD}$. Segment $\overline{AC}$ intersects segments $\overline{EF}$ and $\overline{EG}$ at points H and J, respectively. If the area of rectangle $ABCD$ is 70, find the area of $\triangle EHJ$.

(**A**) $\boxed{3}$ (**B**) 9 (**C**) $\frac{35}{9}$ (**D**) $\frac{70}{9}$ (**E**) None of these

46. Which of the following (x, y) pairs satisfy the equation $\log_y x - 0.75 \log_y x + \log_y 3x = 2$?

(**A**) $(9, 27)$ (**B**) $(27, 3)$ (**C**) $(81, 9)$ (**D**) $\boxed{(81, 27)}$ (**E**) None of these

47. If $x = a + bi$, and $(2i + 1)x = i + 3$, find $a + b$, where i is the imaginary unit.

(**A**) $-\frac{3}{10}$ (**B**) $\boxed{0}$ (**C**) $\frac{1}{5}$ (**D**) $\frac{4}{3}$ (**E**) None of these

48. For an angle $0 \leq \theta \leq \frac{\pi}{2}$, if $\sin\theta = 3\cos\theta$, what is $\tan\theta$?

(**A**) $\frac{1}{8}$ (**B**) $\frac{1}{3}$ (**C**) $\boxed{3}$ (**D**) 8 (**E**) None of these

49. Suppose a right square pyramid has a square base with side length 4 and a surface area of 40. What is the height of this pyramid?

(**A**) 2 (**B**) 3 (**C**) $\boxed{\sqrt{5}}$ (**D**) $\sqrt{6}$ (**E**) None of these

50. There is one solution (a, b) satisfying the system of equations

$$\begin{cases} \log x^2 y^3 = 4 \\ \log \frac{x}{y} = 2 \end{cases}$$

Find a.

(**A**) 2 (**B**) 4 (**C**) 20 (**D**) $\boxed{100}$ (**E**) None of these