The 42nd Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

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ALGEBRA II EXAMINATION

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E. A calculator is NOT permitted.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Find the sum of the x-components of the x-intercepts of the line given by $2x + 3y = 5$ and the line perpendicular to it through the point $(1, 1)$.

(**A**) $\frac{7}{6}$ (**B**) $\boxed{\frac{17}{6}}$ (**C**) $\frac{23}{6}$ (**D**) $\frac{25}{6}$ (**E**) None of these

2. If (a, b) is a solution to the system below, find the smallest possible value of $a + b$.

$$\begin{cases} \frac{1}{x^2} - \frac{3}{y^3} = 1 \\ \frac{2}{x^2} - \frac{1}{y^3} = 7 \end{cases}$$

(**A**) $\boxed{\frac{1}{2}}$ (**B**) $\frac{3}{4}$ (**C**) $\frac{3}{2}$ (**D**) $\frac{5}{4}$ (**E**) None of these

3. The first and fifth terms of an arithmetic sequence are -3 and 17, respectively. Find the 50th term.

(**A**) 193 (**B**) $\boxed{242}$ (**C**) 247 (**D**) 977 (**E**) None of these

4. Simplify $\sqrt{\frac{1}{3}} + 5\sqrt{12} - 2\sqrt{108}$.

(**A**) $-\frac{11\sqrt{3}}{3}$ (**B**) $\boxed{-\frac{5\sqrt{3}}{3}}$ (**C**) $\frac{13\sqrt{3}}{3}$ (**D**) $\frac{25\sqrt{3}}{3}$ (**E**) None of these

5. A jar contains \$6.65 made up of nickels (value of five cents), dimes (value of ten cents), and quarters (value of twenty five cents). There are the same number of nickels as dimes. If there are 70 total coins, how many more dimes are there than quarters?

(**A**) $\boxed{23}$ (**B**) 25 (**C**) 29 (**D**) 31 (**E**) None of these

6. Which of the following pairs (a, x) is a solution to the equation

$$\frac{4a + x}{a + x} - 1 = \frac{12a}{3a + x}?$$

(**A**) $(\sqrt{5}, 3\sqrt{5})$ (**B**) $(3\sqrt{5}, \sqrt{5})$ (**C**) $(-\sqrt{5}, 3\sqrt{5})$ (**D**) $\boxed{(-3\sqrt{5}, \sqrt{5})}$ (**E**) None of these

7. Find the positive real solution to the equation $2x^2 - 3x^{6/5} - 9x^{2/5} = 0$.

(**A**) $\boxed{3\sqrt[4]{3}}$ (**B**) $\frac{3\sqrt[4]{3}}{4}$ (**C**) $\sqrt[5]{81}$ (**D**) $\frac{\sqrt[5]{162}}{2}$ (**E**) None of these

8. Find the sum of the reciprocals of the two solutions to the equation $(2x - 3)(x - 4) = 33$.

(**A**) $\frac{55}{666}$ (**B**) $\frac{39}{74}$ (**C**) $\frac{2}{5}$ (**D**) $\boxed{-\frac{11}{21}}$ (**E**) None of these

9. If a is the largest real solution to the equation $(x^2 - x + 2)^2 + 2(x - 5)(x + 4) = 36$, find $a^3 - 4a + 2$.

(**A**) 2 (**B**) $\boxed{17}$ (**C**) 482 (**D**) 962 (**E**) None of these

10. The reciprocals of two numbers sum to $\frac{1}{120}$. If the two numbers sum to 500, find the positive difference of their reciprocals.

(**A**) 100 (**B**) $\frac{1}{100}$ (**C**) $\boxed{\frac{1}{600}}$ (**D**) $\frac{1}{1200}$ (**E**) None of these

11. What is the distance between the $x-$intercepts of the parabola given by $y = x^2 - 5x - 14$?

(**A**) 5 (**B**) $\boxed{9}$ (**C**) 10 (**D**) 18 (**E**) None of these

12. If (x, y) is a solution to the system of inequalities below, find the minimum value of y.

$$\begin{cases} 3x + 2y \geq 4 \\ 5x - 6y \geq 16 \\ 2x - y \leq 12 \end{cases}$$

(**A**) $\boxed{-4}$ (**B**) $-\frac{3}{2}$ (**C**) 0 (**D**) 2 (**E**) None of these

13. Let $f(x) = |x - 4|$ and $g(x) = \sqrt{(x-2)^2} - 1$. When the solutions of the equation $(g \circ f)(x) = 0$ are written in increasing order, they form an arithmetic progression. What would the next number in that progression be?

(**A**) 5 (**B**) $\boxed{9}$ (**C**) 13 (**D**) 17 (**E**) None of these

14. Jenna makes a New Year's resolution to save money. On the first day, they contribute one dollar, on the second day they contribute two dollars, on the third day they contribute three dollars, etc. This is continued for n days. At the end, Jenna evenly divides the funds into $\frac{n}{2}$ jars. How many dollars are in each jar at the end?

(**A**) n (**B**) $2n$ (**C**) $n - 1$ (**D**) $\boxed{n + 1}$ (**E**) Impossible to determine

15. What is the largest prime factor in the prime factorization of the number 3,427,200?

(**A**) 8 (**B**) 9 (**C**) $\boxed{17}$ (**D**) 21 (**E**) 35

16. Find sum of the squares of all solutions to the equation $x^4 - 4x^3 - 9x^2 + 12x + 18 = 0$.

(**A**) 14 (**B**) $\boxed{34}$ (**C**) 58 (**D**) 94 (**E**) None of these

17. Find the sum of solutions to the equation $\log_2(x)\,(\log_2(x) - 8) = -15$.

(**A**) 8 (**B**) 16 (**C**) 34 (**D**) $\boxed{40}$ (**E**) None of these

18. What is the minimum value of the expression $x - 3\sqrt{x} + 4$?

(**A**) $\frac{3}{2}$ (**B**) $\frac{5}{2}$ (**C**) $\boxed{\frac{7}{4}}$ (**D**) $\frac{9}{4}$ (**E**) None of these

19. For how many values of k does the equation $x^3 + 3x^2 + 3x + k = 0$ have a single real solution?

(**A**) 0 (**B**) 1 (**C**) 3 (**D**) $\boxed{\text{infinitely many}}$ (**E**) None of these

20. A Catalan number is any number defined by the sequence $u_n = \dfrac{(2n)!}{(n+1)!n!}$. Find u_5, the fifth Catalan number.

(**A**) $\dfrac{1}{360}$ (**B**) $\dfrac{1}{6}$ (**C**) $\boxed{42}$ (**D**) 1008 (**E**) None of these

21. Which of the following is a square root of $4a^2 + 4ax^2 - 4abx + x^4 + b^2x^2 - 2bx^3$?

(**A**) $2a - bx - x^2$ (**B**) $2a + bx + x^2$ (**C**) $2a + bx - x^2$ (**D**) $\boxed{2a - bx + x^2}$ (**E**) None of these

22. Which of the following is the multiplicative inverse of the matrix $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$?

(**A**) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ -1 & \frac{1}{5} \end{bmatrix}$ (**B**) $\begin{bmatrix} -2 & -3 \\ 1 & -5 \end{bmatrix}$ (**C**) $\begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix}$ (**D**) $\begin{bmatrix} -2 & -1 \\ 3 & -5 \end{bmatrix}$ (**E**) $\boxed{\text{None of these}}$

23. The equation $x - \sqrt[3]{3x+2} = 0$ has two solutions a and b. Find $a + b$.

(**A**) -2 (**B**) -1 (**C**) 0 (**D**) $\boxed{1}$ (**E**) None of these

24. The function $f(x) = 3x^2 - kx + 4$ will have two positive real roots if:

(**A**) $\boxed{k > 4\sqrt{3}}$ (**B**) $|k| < 4/3$ (**C**) $k < 12$ (**D**) $k = 0$ (**E**) None of these

25. What is the value of the function $f(x) = \dfrac{3x^2 - x - 2}{6x^2 - 5x - 1}$ at $x = 1$.

(**A**) $-\dfrac{2}{7}$ (**B**) $\dfrac{5}{7}$ (**C**) $\dfrac{1}{5}$ (**D**) $\boxed{\text{Undefined}}$ (**E**) None of these

26. Define $f(x) = 2x^2 + 6x - 7$ on the interval $[-1.5, \infty)$. What is the value of $f^{-1}(13)$?

(**A**) 5 (**B**) $\boxed{2}$ (**C**) $\dfrac{1}{409}$ (**D**) $\dfrac{1}{2023}$ (**E**) None of these

27. Which of the following are rational numbers?

I. -5 II. $\dfrac{3}{2}$ III. $\sqrt{2}$ IV. $0.\overline{123}$

(**A**) II only (**B**) III only (**C**) I, II only (**D**) II, IV only (**E**) $\boxed{\text{I, II, IV only}}$

28. Two fair six-sided dice are rolled. Given that at least one of the die showed an even number, what is the probability that the product of the two numbers shown by the dice is divisible by 4?

(**A**) $\dfrac{2}{3}$ (**B**) $\dfrac{3}{4}$ (**C**) $\boxed{\dfrac{5}{9}}$ (**D**) $\dfrac{5}{12}$ (**E**) None of these

29. In the equation $x^2 + 3hx = 1$ the sum of the squares of the solutions is 3. Find $|h|$.

(**A**) $\dfrac{2}{9}$ (**B**) $\boxed{\dfrac{1}{3}}$ (**C**) $\dfrac{\sqrt{2}}{3}$ (**D**) $\dfrac{\sqrt{6}}{3}$ (**E**) None of these

30. Use the fact that $\log_{10} 2 \approx 0.301$ to find the approximate value of $\log_{10} 50$.

(**A**) 0.7525 (**B**) 1.301 (**C**) 1.505 (**D**) $\boxed{1.699}$ (**E**) None of these

31. Let $f(x) = 2x^3 - 6x^2 + 6x - 18$ and $g(x) = x^2 + 4$. When $x = 80$, what is the positive remainder when $f(x)$ is divided by $g(x)$?

(**A**) 18 (**B**) 154 (**C**) $\boxed{6250}$ (**D**) 6404 (**E**) None of these

32. Find the value of $\sqrt{\dfrac{(5+i)(2-2i)}{3-2i}}$ where i is the imaginary unit.

(**A**) $\boxed{2}$ (**B**) -2 (**C**) $2+2i$ (**D**) $2-2i$ (**E**) None of these

33. If $f(x)$ is an even function, which of the following must also be even?

I. $f(5x)$ II. $f(x)+5$ III. $-5f(x)$ IV. $f(x-5)$

(**A**) I, III only (**B**) II, IV only (**C**) $\boxed{\text{I, II, III only}}$ (**D**) I, II, III, IV (**E**) None of these

34. Let $f(x) = x^5 - 28x^3 + 75x$. The function $f(x)$ has two non-zero rational roots, one of which is $x = 5$ and the other we will call a. Find $a^2 + 3a + 2$.

(**A**) -38 (**B**) 0 (**C**) 6 (**D**) $\boxed{12}$ (**E**) None of these

35. Solve for x if $2^{3x+2} + 8^x = \dfrac{5}{16}$.

(**A**) $\boxed{-\dfrac{4}{3}}$ (**B**) $-\dfrac{3}{4}$ (**C**) -2 (**D**) $-\dfrac{1}{2}$ (**E**) None of these

36. For $f(x)$ defined below, find the sum of all values for which $f(x) = 1$.

$$f(x) = \begin{cases} x^2 + 6x + 6 & x \le 0 \\ 2x - 5 & 0 < x < 3 \\ \sqrt{x-3} & x \ge 3 \end{cases}$$

(**A**) $\boxed{-2}$ (**B**) -1 (**C**) 1 (**D**) 3 (**E**) None of these

37. One bag contains 5 white and 3 red balls, and a second bag contains 4 white and 5 red balls. From one of them, chosen at random, two balls are drawn without replacement. Find the probability that the balls are different colors.

(**A**) $\dfrac{9}{34}$ (**B**) $\dfrac{37}{72}$ (**C**) $\dfrac{72}{289}$ (**D**) $\boxed{\dfrac{275}{504}}$ (**E**) None of these

38. The expression $\dfrac{11x^2 - 3}{(x+1)^2(x-3)}$ can be decomposed as $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-3}$. Find $A + B + C$.

(**A**) $\dfrac{27}{5}$ (**B**) $\dfrac{69}{5}$ (**C**) $\boxed{9}$ (**D**) -11 (**E**) None of these

39. Let $p(x)$ be a polynomial that has $x-1$, x^2-1, x^3-1 and x^4-1 as factors. What is the smallest possible degree of $p(x)$?

(**A**) $\boxed{6}$ (**B**) 7 (**C**) 8 (**D**) 10 (**E**) None of these

40. How many integers are in the solution set to $|x^2-2x-2| > |x^2-2x+2|$?

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 4 (**E**) None of these

41. Which of the following is equal to $\sqrt{17+12\sqrt{2}}+\sqrt{17-12\sqrt{2}}$?

(**A**) 5 (**B**) $\boxed{6}$ (**C**) $2\sqrt{17}$ (**D**) $\sqrt{29}$ (**E**) None of these

42. Which of the following polynomials has a graph which does not contain points in the second quadrant?

(**A**) x^2+7x-4 (**B**) $-x^2+6$ (**C**) x^4-3x^2+x (**D**) $\boxed{-x^4+2x^3-1}$ (**E**) $-4x^5+2x-7$

43. The sum of three consecutive terms in a geometric sequence is 38 and their product is 1728. Find the difference between the largest and smallest of those three numbers.

(**A**) $\boxed{10}$ (**B**) 11 (**C**) 12 (**D**) 14 (**E**) None of these

44. Solve for x in the equation $\dfrac{3x+4}{12}-\dfrac{2x-3}{9}=\dfrac{11}{6}$.

(**A**) $\boxed{42}$ (**B**) 65 (**C**) 66 (**D**) 90 (**E**) None of these

45. Simplify and factor the following: $\dfrac{r+\dfrac{t^2}{r-2t}}{\dfrac{r-t}{r^2-4t^2}}$.

(**A**) $(r+t)(r+2t)$ (**B**) $(r-t)(r-2t)$ (**C**) $(r+t)(2t-r)$

(**D**) $(r+t)(r-2t)$ (**E**) $\boxed{(r-t)(r+2t)}$

46. Define an operation & on the integers by $a\&b=\sqrt{|a^2-b^2|}$. Find 27&45.

(**A**) $9\sqrt{22}$ (**B**) $9\sqrt{34}$ (**C**) 18 (**D**) $\boxed{36}$ (**E**) None of these

47. What is the coefficient of the x term of the least degree polynomial that passes through the points $(1,1)$, $(0,-1)$, and $(2,4)$?

(**A**) -3 (**B**) $-\dfrac{5}{2}$ (**C**) $\dfrac{1}{2}$ (**D**) $\boxed{\dfrac{3}{2}}$ (**E**) None of these

48. If $f(2x)=x^2-5x+3$, what is $f(7)$?

(**A**) -46 (**B**) $\boxed{-\dfrac{9}{4}}$ (**C**) 17 (**D**) 129 (**E**) None of these

49. If $x^2 + 2x + 5$ is a factor of $x^4 + px^2 + q$, then what is $p + q$?

(**A**) 10 (**B**) 19 (**C**) 26 (**D**) $\boxed{31}$ (**E**) None of these

50. Trent is preparing a salad. He can make his salad with just lettuce, or he can make it with lettuce and any of the following toppings: cucumbers, tomatoes, onions, carrots, mushrooms, croutons, and/or cheese. How many variations of the salad are possible?

(**A**) 49 (**B**) 64 (**C**) $\boxed{128}$ (**D**) 256 (**E**) None of these