The 41st Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

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ALGEBRA II EXAMINATION

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. The quartic polynomial $3x^4 - 8x^3 - 50x^2 - 57x - 18$ can be factored as $(x^2 - 5x - 6)(ax^2 + bx + c)$. Find $a + b + c$.

(**A**) -23 (**B**) 7 (**C**) 13 (**D**) 27 (**E**) None of these

2. Two roots of the polynomial $3x^3 + ax^2 - 5x + 10$ are r and $-r$ for r a real number. What is the value of a?

(**A**) -6 (**B**) 6 (**C**) -4 (**D**) 4 (**E**) None of these

3. Let $f(x) = x^3 + 4$ and $g(x) = x^2 - 1$, with $g(x)$ restricted to the domain $[0, \infty)$. Find the value of $(g \circ f)^{-1}(8)$.

(**A**) -1 (**B**) 1 (**C**) $\sqrt[3]{7}$ (**D**) $3\sqrt[3]{4}$ (**E**) None of these

4. Find the value of $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots + 19^2 - 20^2$.

(**A**) -13175 (**B**) -210 (**C**) 1330 (**D**) 2870 (**E**) None of these

5. The graph of the function $f(x) = x^5 - 4x^3 - 8$ crosses the x-axis exactly once. In which of the following intervals does it cross the x-axis?

(**A**) $[-4, -2]$ (**B**) $[-2, 0]$ (**C**) $[0, 2]$ (**D**) $[2, 4]$ (**E**) None of these

6. A history final exam has 25 questions. If the questions were all equally weighted, Norman would have earned a 72%. However, the first twenty questions are worth three points each, and the final five questions are worth eight points each. If Norman got a 64% on the test with this point distribution, how many eight point questions did Norman get correct?

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) None of these

7. The graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ passes through the points $(6, 1598)$, $(-3, -292)$, $(0, -4)$ and $(4, 464)$, with a, b, c, and d all integer coefficients. What is the value of the product $abcd$?

(**A**) -945 (**B**) -362 (**C**) 525 (**D**) 810 (**E**) None of these

8. If a and b are the solutions to the equation $(2x + 3)(3x - 1) = 7$, find $|a| + |b|$.

(**A**) $\frac{8}{3}$ (**B**) $\frac{14}{3}$ (**C**) $-\frac{7}{6}$ (**D**) $\frac{17}{6}$ (**E**) None of these

9. Mrs. Driskell teaches three sections of intermediate algebra, and gives a common final to all three sections. The first section has 32 students and had an average on the final of 60%. The second section has 38 students and had an average of 70%, and the third section has 30 students and had an average of 62%. What is the overall average of all of Mrs. Driskell's students on the final?

(**A**) 63.6% (**B**) 64% (**C**) 64.4% (**D**) 64.8% (**E**) None of these

10. Find the smallest integer value of $|x|$ for which (x, y) is a solution to $\frac{2x + 3}{y - 2} = 7$ and y is also an integer.

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) None of these

11. The value of $\log_{10}(172)$ rounded to four decimal places is 2.2355. Find the value of $\log_{10}(0.172)$ rounded to four decimal places.

(**A**) -1.7645 (**B**) -0.7645 (**C**) 0.0022 (**D**) 0.0224 (**E**) 0.2355

12. Let a and b be real numbers, with $a \neq 0$. The quadratic equation $a^2x^2 - 4abx - 5b^2 = 0$ has a unique real solution when:

(**A**) $a = 1$ (**B**) $b = 0$ (**C**) $a + b = 0$ (**D**) $\dfrac{5b^2}{a^2} = 1$ (**E**) None of these

13. Consider the system of equations below

$$\begin{cases} x - y = 4 \\ xy = 6 \end{cases}$$

If (a, b) is a solution to the system with $a, b > 0$, find $a + b$.

(**A**) $2\sqrt{10}$ (**B**) $4\sqrt{10}$ (**C**) $8 + 2\sqrt{10}$ (**D**) $8 + 4\sqrt{10}$ (**E**) None of these

14. The equation $x^3 + 27 = 0$ has one real solution r and two complex solutions of the form $a + bi$ and $c + di$. Find $a^2 + b^2 + c^2 + d^2$.

(**A**) $\dfrac{9}{2}$ (**B**) 9 (**C**) 18 (**D**) 27 (**E**) None of these

15. Find the square root of the polynomial $x^4 - 4x^3 + 10x^2 - 12x + 9$.

(**A**) $x^2 - 5x + 3$ (**B**) $x^2 - 2x + 3$ (**C**) $x^2 + x - 3$ (**D**) $x^2 + 3x - 3$ (**E**) $x^2 + 4x + 3$

16. Let $f(x) = 2x^2 - 3x + 1$. If the graph of a function $y = g(x)$ is formed by shifting the graph of $f(x)$ 2 units up and 3 units to the right, then $g(x)$ is given by

(**A**) $2x^2 + 5x + 6$ (**B**) $2x^2 + 3x + 12$ (**C**) $2x^2 + 9x + 12$ (**D**) $2x^2 - 15x + 30$ (**E**) None of these

17. Define values m and n in terms of variables a and x by $m = 2a^3 - 2ax^2$ and $n = 2a^3 + 4a^2x + 2ax^2$. When $a = 40$ and $x = 35$, what is the greatest common divisor of m and n?

(**A**) 5 (**B**) 2,000 (**C**) 6,000 (**D**) 30,000 (**E**) None of these

18. Find the sum of all values of x for which the equation $\left(\sqrt{|x+2|} + \sqrt{|2x-1|}\right)^2 = |x+2| + |1-2x|$ is true.

(**A**) $-\dfrac{5}{2}$ (**B**) $-\dfrac{3}{2}$ (**C**) $\dfrac{1}{2}$ (**D**) This is true for infinitely many values of x (**E**) None of these

19. The system of equations below has a unique solution (a, b). Find the value of $3a + 2b$.

$$\begin{cases} 4^x 8^y = 32 \\ x - y = 10 \end{cases}$$

(**A**) 5 (**B**) $\dfrac{15}{2}$ (**C**) 15 (**D**) 25 (**E**) None of these

20. The maximum number of intersection points possible between the graphs of a quadratic function and a cubic function is:

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

21. Find the product of the largest and smallest solutions of the equation $(2x^2+4x-3)^2 = 9-4x-2x^2$.

(**A**) -6 (**B**) -3 (**C**) $-\frac{5}{2}$ (**D**) 0 (**E**) None of these

22. What is the value of the last non-zero digit in the decimal expansion of 1.2^7?

(**A**) 2 (**B**) 4 (**C**) 6 (**D**) 8 (**E**) None of these

23. Consider the quadratic function $f(x) = 6x^2+ax+4$. Which of the following values **cannot** be a zero of $f(x)$ for any integer value of a?

(**A**) $-\frac{2}{3}$ (**B**) $-\frac{1}{6}$ (**C**) $-\frac{1}{3}$ (**D**) $\frac{1}{2}$ (**E**) $\frac{3}{2}$

24. A jar contains 10 Red, 8 Purple, and 7 Blue marbles. If you draw two marbles without replacement, what is the probability you get one red and one blue?

(**A**) $\frac{14}{125}$ (**B**) $\frac{83}{120}$ (**C**) $\frac{7}{60}$ (**D**) $\frac{7}{30}$ (**E**) None of these

25. For $f(x)$ defined as the piecewise function below, find the sum of all real values of x for which $f(x) = 0$.

$$f(x) = \begin{cases} |x+3| & \text{for } x \le -1 \\ x^2+3x-4 & \text{for } -1 < x < 2 \\ \log_5(\frac{1}{5}x) & \text{for } x \ge 2 \end{cases}$$

(**A**) -6 (**B**) -2 (**C**) -1 (**D**) 3 (**E**) None of these

26. Find the average of the first ten terms in the geometric series which begins $3, 6, 12, 24, ...$

(**A**) 153.6 (**B**) 202.2 (**C**) 306.9 (**D**) 460.8 (**E**) None of these

27. Define an operation # for positive real numbers by $a\#b = \frac{ab}{a+b}$. What is the value of $8\#(8\#8)$?

(**A**) $\frac{1}{2}$ (**B**) $\frac{8}{3}$ (**C**) 4 (**D**) 16 (**E**) None of these

28. If the values of A and B make the equation $\frac{35x-29}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$ true for all values of x for which it is defined, find $A+B$.

(**A**) 6 (**B**) 29 (**C**) 35 (**D**) 47 (**E**) None of these

29. If $a+b=4$, and $a^2+b^2=12$, then what is a^4+b^4?

(**A**) 112 (**B**) 136 (**C**) 144 (**D**) 256 (**E**) None of these

30. If a line passes through the points $(1, 2)$ and $(-0.5, 1)$, what is the distance between the x and y intercepts of that line?

(**A**) $\frac{\sqrt{13}}{6}$ (**B**) $\frac{\sqrt{13}}{2}$ (**C**) $\frac{2\sqrt{13}}{3}$ (**D**) $\frac{3\sqrt{13}}{2}$ (**E**) None of these

31. If $f(x) = \dfrac{x^{26} + x^{24} + 2x^{22}}{x - 1}$, find $f(i)$, where i is the imaginary unit.

(**A**) $-1 - i$ (**B**) $-1 + i$ (**C**) $1 - i$ (**D**) $1 + i$ (**E**) None of these

32. If $\begin{bmatrix} 10 & 1 \\ -6 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find $a + b + c + d$.

(**A**) -4 (**B**) $-\frac{3}{2}$ (**C**) $\frac{19}{2}$ (**D**) 6 (**E**) None of these

33. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, for $-1 < x < 1$, write $f\left(\dfrac{3x + x^3}{1 + 3x^2}\right)$ in terms of $f(x)$.

(**A**) $f(x^3)$ (**B**) $[f(x)]^3$ (**C**) $f(x + 3)$ (**D**) $f(3x)$ (**E**) $3f(x)$

34. A harshad number is an integer number which is divisible by the sum of its digits. Which of the following is NOT a harshad number?

(**A**) 2022 (**B**) 2023 (**C**) 2024 (**D**) 2025 (**E**) 2026

35. A middle school basketball player makes a free throw with a probability of 0.6. Assuming each free throw is an independent event, what is the probability the player makes zero of their next three free throws?

(**A**) 0.064 (**B**) 0.216 (**C**) 0.784 (**D**) 0.936 (**E**) None of these

36. What is the shortest distance from the point $(6, 5)$ to the line $2x + 3y = 1$?

(**A**) $\sqrt{13}$ (**B**) $2\sqrt{13}$ (**C**) $4\sqrt{13}$ (**D**) $6\sqrt{13}$ (**E**) None of these

37. Find the median of the solutions to the equation $3x^3 + 5x^2 - 6x - 10 = 0$.

(**A**) $-\sqrt{2}$ (**B**) $\sqrt{2}$ (**C**) $-\frac{5}{9}$ (**D**) $-\frac{3}{5}$ (**E**) None of these

38. Find the number of real solutions to the equation $\sqrt[2]{x+1} = \sqrt[3]{3x-1}$.

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) None of these

39. Given that the graph of the parabola $y = ax^2 - 32x + 4$ has an axis of symmetry of $x = 8$, find the value of a.

(**A**) $\frac{63}{64}$ (**B**) $\frac{3}{2}$ (**C**) 2 (**D**) 4 (**E**) None of these

40. If $\dfrac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}} = 41$, then what is $\dfrac{x}{y}$?

(**A**) $-\frac{20}{21}$ (**B**) $\frac{1}{20}$ (**C**) $\frac{20}{21}$ (**D**) 20 (**E**) None of these

41. Which of the following function(s) has a unique inverse?

I. $f(x) = x^2 + 1$ II. $f(x) = e^x$ III. $f(x) = x^3 - 2$

(**A**) Only I (**B**) Only II (**C**) Only III (**D**) Both II and III but not I (**E**) All of I, II, and III

42. Two standard six-sided dice are rolled. Find the probability that you roll a sum that is an even number, given that the product is an even number.

(**A**) $\frac{1}{3}$ (**B**) $\frac{2}{5}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{1}{6}$ (**E**) None of these

43. What is the smallest positive integer value of b for which the function $f(x) = x^2 + bx + 5$ has real zeros?

(**A**) 3 (**B**) 4 (**C**) 5 (**D**) 6 (**E**) None of these

44. If $f(x)$ is an odd function, which of the following is NOT an even function?

(**A**) $f(|x|)$ (**B**) $|f(x)|$ (**C**) $f(x^2)$ (**D**) $[f(x)]^2$ (**E**) $f(-x)$

45. Find the remainder when $f(x) = (x+4)(x^2-7) + (x+2)(2x-4) - (x+1)(x^3+2)(2x-1)$ is divided by $x - 1$.

(**A**) -42 (**B**) -34 (**C**) -32 (**D**) -24 (**E**) None of these

46. Find the smallest solution of the equation $2^{x-1} + 2^{-(x+1)} = 4$.

(**A**) $\log_2(4 - \sqrt{17})$ (**B**) $\log_2(4 - \sqrt{15})$ (**C**) $\log_2(4 - \sqrt{14})$ (**D**) $\log_2(4 - \sqrt{13})$ (**E**) None of these

47. Which of the following (x, y) ordered pairs is a solution to the inequality $\dfrac{(3x+3y)^8(2x-2y)^7}{6^7(x^2-y^2)^8} > 10$?

(**A**) $\left(\frac{3}{5}, \frac{1}{4}\right)$ (**B**) $\left(\frac{3}{2}, 1\right)$ (**C**) $\left(0, -\frac{2}{3}\right)$ (**D**) $\left(1, \frac{4}{5}\right)$ (**E**) None of these

48. The function $f(x) = x^4 + 4x^3 + 10x^2 + 12x + 12$ has a minimum value of $y = b$ occurring at $x = a$. Find $a + b$.

(**A**) 3 (**B**) 6 (**C**) 10 (**D**) 12 (**E**) None of these

49. A particular function $f(n)$ defined on the positive integers has a value of $3n + 1$ if n is odd, and $\frac{n}{2}$ if n is even. That is,

$$f(n) = \begin{cases} 3n+1 & \text{for } n \text{ odd} \\ n/2 & \text{for } n \text{ even} \end{cases}$$

What is the value of $f(f(f(f(f(10)))))$?

(**A**) 2 (**B**) 4 (**C**) 16 (**D**) 25 (**E**) None of these

50. The equation $2x + 3y - 4z = 5$ has infinitely many integer solutions. Which of the following cannot be a value of y for any integer values of x and z?

(**A**) 2 (**B**) 3 (**C**) 5 (**D**) 9 (**E**) None of these