The 38th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 23, 2019 at Regional Testing Centers
Second Round: April 6, 2019 at The University of Alabama at Birmingham

COMPREHENSIVE EXAM

Construction of this test directed
by
Miranda Bowie and Ashley Johnson, The University of North Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

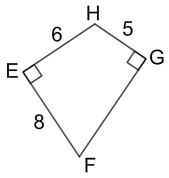
- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. A local movie theater sells a small popcorn for \$4, a medium for \$6, and a large for \$8. Last year, Miranda bought a total of 14 containers of popcorn, costing a total of \$78. If she bought two fewer smalls than mediums and larges combined, how many containers of large popcorn did she buy?

(**A**) $\boxed{3}$ (**B**) 4 (**C**) 5 (**D**) 6 (**E**) None of these

2. In the figure shown, $\square EFGH$ is a quadrilateral with right angles at vertices E and G, $EF = 8$, $EH = 6$, and $HG = 5$. Find GF.



(**A**) 3 (**B**) 4 (**C**) $\boxed{5\sqrt{3}}$ (**D**) $5\sqrt{5}$ (**E**) None of these

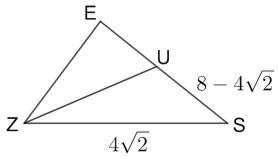
3. Find the sum of all values on the interval $[0, 2\pi)$ for which $2\cos^3 x + \cos^2 x - 2\cos x - 1 = 0$.

(**A**) $\frac{5\pi}{3}$ (**B**) 2π (**C**) $\boxed{3\pi}$ (**D**) $\frac{11\pi}{3}$ (**E**) None of these

4. Find the sum of the values x_1 and x_2 satisfying the equation $\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

(**A**) -2 (**B**) $-\frac{3}{2}$ (**C**) $\boxed{-1}$ (**D**) $-\frac{1}{2}$ (**E**) None of these

5. Triangle ZES shown is isosceles with $ZE = ES$, point U is on $\overline{ES}$ such that $\overrightarrow{ZU}$ bisects angle $\angle EZS$. If $ZS = 4\sqrt{2}$, and $US = 8 - 4\sqrt{2}$, find the length of $\overline{EU}$.



(**A**) $\boxed{4\sqrt{2} - 4}$ (**B**) 4 (**C**) $8 - 4\sqrt{2}$ (**D**) 8 (**E**) None of these

6. Write $\sqrt{16 + 30i}$ in $a + bi$ form with $a, b \geq 0$.

(**A**) $4 + \sqrt{30}\, i$ (**B**) $15 + i$ (**C**) $4\sqrt{2} + \sqrt{2}\, i$ (**D**) $45 + \frac{1}{3}i$ (**E**) $\boxed{\text{None of these}}$

7. The graph of $f(x) = x^5 - 4x^4 + 3x^2 + 6$ has one root less than 0. On which of the following intervals does it lie?

(**A**) $(-5, -4)$ (**B**) $(-4, -3)$ (**C**) $(-3, -2)$ (**D**) $\boxed{(-2, -1)}$ (**E**) $(-1, 0)$

8. Find the sum of the absolute values of all solutions to the equation $(4x + 3)(x - 1) = 15$.

(**A**) $\frac{7}{4}$ (**B**) $\boxed{\frac{17}{4}}$ (**C**) 5 (**D**) 19 (**E**) None of these

9. In parallelogram $ABCD$, the measure of angle $\angle BCD$ is $56°$. What is the measure of angle $\angle CDA$?

(**A**) $34°$ (**B**) $56°$ (**C**) $72°$ (**D**) $112°$ (**E**) $\boxed{\text{None of these}}$

10. Find the area of the region in the x, y-plane which satisfies $y \geq 0$, $x + y \leq 3$ and $y \leq 2x$.

(**A**) 2 (**B**) 2.5 (**C**) $\boxed{3}$ (**D**) 4.5 (**E**) None of these

11. Find the median of the solution set to the equation $\left(x - \frac{5}{x}\right)^2 - 2x + \frac{10}{x} = 24$.

(**A**) 0 (**B**) 1 (**C**) $\dfrac{2-\sqrt{14}}{2}$ (**D**) $\boxed{\dfrac{4-\sqrt{14}}{2}}$ (**E**) None of these

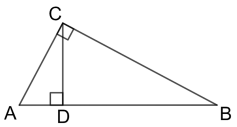
12. Find the sum of the coefficients of the quotient $(3x^3 - 4x^2y + 5xy^2 + 6y^3) \div (x^2 - 2xy + 3y^2)$.

(**A**) -7 (**B**) -1 (**C**) 3 (**D**) $\boxed{5}$ (**E**) None of these

13. The graph of $9x^2 - 30xy + 25y^2 = 0$ is best described as

(**A**) an ellipse (**B**) a circle (**C**) a parabola (**D**) a hyperbola (**E**) $\boxed{\text{a line}}$

14. In the right triangle ABC shown, $\angle ACB$ is a right angle, D is on $\overline{AB}$, $\overline{CD} \perp \overline{AB}$, $AD = 4$, and $CD = 6$. Find the area of triangle ABC in square units.



(**A**) 36 (**B**) $\boxed{39}$ (**C**) 45 (**D**) 48 (**E**) None of these

15. Let $f(x) = x^2 + 2$, and $g(x)$ be a linear function. Find the x-coordinate of the x-intercept of $g(x)$ if $(f \circ g)(x) = 4x^2 - 28x + 51$.

(**A**) $\boxed{3.5}$ (**B**) -3.5 (**C**) 7 (**D**) -7 (**E**) None of these

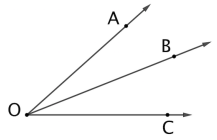
16. Find the product of all solutions to the equation $\log(\sqrt[3]{x}) = \sqrt{\log x}$.

(**A**) 0 (**B**) 10^3 (**C**) 10^6 (**D**) $\boxed{10^9}$ (**E**) None of these

17. Find the value of $\log_2\left(2^1 \cdot 4^2 \cdot 8^3 \cdot \cdots \cdot 512^9\right)$.

(**A**) 90 (**B**) $\boxed{285}$ (**C**) 2019 (**D**) 3321 (**E**) None of these

18. In the figure given, ray $\overrightarrow{OB}$ bisects angle $\angle AOC$. Given the measure of angle $\angle AOB$ is $3x + 12$, and the measure of angle $\angle BOC$ is $18x - 18$, find the supplement of the measure of angle $\angle AOC$.



(**A**) 42° (**B**) 54° (**C**) 138° (**D**) $\boxed{144°}$ (**E**) None of these

19. Find the sum of all solutions to the equation $x|x - 1| - 3|2 - 2x| = 0$.

(**A**) -6 (**B**) -5 (**C**) 6 (**D**) $\boxed{7}$ (**E**) None of these

20. Among all points which lie on the line $y = 5x - 3$, find the value of x which minimizes the value of $\sqrt{x^2 + y^2}$.

(**A**) 0 (**B**) $\boxed{\dfrac{15}{26}}$ (**C**) $\dfrac{5}{8}$ (**D**) $\dfrac{3}{5}$ (**E**) None of these

21. The consecutive angles of a pentagon form an arithmetic sequence. If the smallest angle measures 60 degrees, what is the measure of the largest angle?

(**A**) 96° (**B**) $\boxed{156°}$ (**C**) 180° (**D**) 300° (**E**) None of these

22. What is the value of $\sin(\frac{\pi}{6}) + \sin(\frac{2\pi}{6}) + \sin(\frac{3\pi}{6}) + \cdots + \sin(\frac{2019\pi}{6})$?

(**A**) $\frac{1+\sqrt{3}}{2}$ (**B**) $\boxed{\frac{3+\sqrt{3}}{2}}$ (**C**) $\frac{1+\sqrt{2}}{2}$ (**D**) $\frac{2+\sqrt{2}}{2}$ (**E**) None of these

23. How many four digit numbers with a 5 in the thousands place, and a 4 in the tens place are divisible by 15?

(**A**) 3 (**B**) 5 (**C**) $\boxed{7}$ (**D**) 10 (**E**) None of these

24. Find the sum of all values of k for which $x^2 + kx + 3 = k$ has only one solution for x.

(**A**) $\boxed{-4}$ (**B**) -3 (**C**) 0 (**D**) 6 (**E**) None of these

25. If $\sin\theta = \dfrac{x}{\sqrt{x^2+4}}$ for some $\theta \in (0, \pi/2)$, find an expression for $\sec\theta$.

(**A**) $\boxed{\frac{\sqrt{x^2+4}}{2}}$ (**B**) $\frac{\sqrt{x^2+4}}{x}$ (**C**) $\frac{2}{\sqrt{x^2+4}}$ (**D**) $\frac{x}{2}$ (**E**) $\frac{2}{x}$

26. The area of a rhombus is 130 square units, and one diagonal has length 13 units. What is the length of the other diagonal?

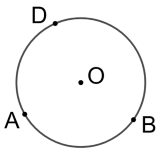
(**A**) 5 (**B**) 10 (**C**) $\boxed{20}$ (**D**) 30 (**E**) None of these

27. Functions f and g are both one-to one where $f(-2) = 5$, $f(-1) = 3$, $f(0) = 6$, $f(1) = -5$, $f(2) = 4$, and $g(-2) = 0$, $g(-1) = 7$, $g(0) = 3$, $g(1) = 2$, $g(2) = -1$. Find the value of $(f \circ g^{-1})(2)$.

(**A**) $\boxed{-5}$ (**B**) 3 (**C**) 4 (**D**) 6 (**E**) None of these

28. In the given circle, chord $\overline{AB}$ has length 12 and the measure of arc $\widehat{ADB}$ is $240°$. What is the radius of the circle?

(**A**) 4 (**B**) 6 (**C**) $2\sqrt{3}$ (**D**) $\boxed{4\sqrt{3}}$ (**E**) None of these



29. Let $\phi(n)$ represent the number of positive integers k less than n with $\gcd(n, k) = 1$. Find $\phi(15)$.

(**A**) $\boxed{8}$ (**B**) 10 (**C**) 12 (**D**) 14 (**E**) None of these

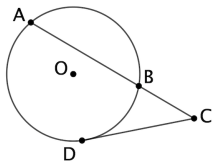
30. Let $f(x) = \dfrac{x+2}{x-4}$. Find the vertical asymptote of the inverse function $f^{-1}(x)$.

(**A**) $x = -2$ (**B**) $x = -1$ (**C**) $x = -\frac{1}{2}$ (**D**) $\boxed{x = 1}$ (**E**) None of these

31. Find the value of n for which $n^{\log_{12}(17)} = 289$

(**A**) 2 (**B**) $\sqrt{12}$ (**C**) $\sqrt{17}$ (**D**) $\boxed{144}$ (**E**) None of these

32. In the figure, points A, B, and D lie on the circle centered at O, with $\overleftrightarrow{CD}$ tangent to the circle at point D. If $CD = 12$ and $BC = 8$, find the value of AB.

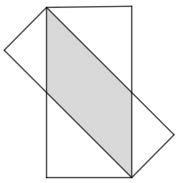


(**A**) 4 (**B**) $\boxed{10}$ (**C**) 12 (**D**) 18 (**E**) None of these

33. Find the slope of the line through the vertex of the parabola given by $y = x^2 - 4x + 5$ and the center of the circle given by $(x+1)^2 + (y-3)^2 = 9$.

(**A**) $\boxed{-\frac{2}{3}}$ (**B**) $-\frac{20}{3}$ (**C**) 4 (**D**) -14 (**E**) None of these

34. Two congruent rectangles each measuring 6 inches by 14 inches overlap as shown. What is the area of the shaded region in square inches?



(**A**) $\frac{240}{7}$ (**B**) 42 (**C**) 48 (**D**) $\boxed{\frac{348}{7}}$ (**E**) None of these

35. A line parallel to $2x + 5y - 8 = 0$ contains points $(3, 7)$ and $(k, 3)$. Find k.

(**A**) -7 (**B**) $\frac{7}{5}$ (**C**) $\frac{23}{5}$ (**D**) $\boxed{13}$ (**E**) None of these

36. Let $f(x)$ be piecewise defined as

$$f(x) = \begin{cases} -x^3 & \text{for } x < -2 \\ x^2 + 4 & \text{for } -2 \le x \le 2 \\ x^3 & \text{for } x > 2 \end{cases}$$

Then the function $f(x)$ is

(**A**) $\boxed{\text{Even}}$ (**B**) Odd (**C**) Neither even nor odd

(**D**) Both even and odd (**E**) Not enough information

37. What is the exact value of $\cos^2(105^\circ) - \sin^2(105^\circ)$?

(**A**) $\boxed{-\frac{\sqrt{3}}{2}}$ (**B**) $-\frac{1}{2}$ (**C**) 1 (**D**) $\frac{\sqrt{2}}{2}$ (**E**) None of these

38. Find the product of all solutions to the equation $4^x(8^{x^2}) = 64(2^{x^3})$.

(**A**) $\boxed{-6}$ (**B**) -1 (**C**) 0 (**D**) 1 (**E**) None of these

39. The circle $x^2 + y^2 = 4$ has a tangent line through $(3, 1)$ with the point of tangency (a, b) in Quadrant IV. Find the sum $a + b$.

(**A**) 0 (**B**) 2 (**C**) $\boxed{\frac{8 - 2\sqrt{6}}{5}}$ (**D**) $\frac{4\sqrt{10}}{5}$ (**E**) None of these

40. Find the 2019^{th} term of the geometric progression whose first, second, and third terms are $\sqrt[3]{9}$, $3\sqrt[3]{3}$, and 9.

(**A**) 3^{673} (**B**) $\boxed{3^{1346}}$ (**C**) 3^{2019} (**D**) 3^{2692} (**E**) None of these

41. Triangle $\triangle ABC$ has side lengths 5, 6, and 7 with an area of $6\sqrt{6}$. Find the sum of the three altitudes of the triangle.

(**A**) $\boxed{\frac{214\sqrt{6}}{35}}$ (**B**) $\frac{35\sqrt{6}}{2}$ (**C**) 18 (**D**) 36 (**E**) None of these

42. Which of the following are true for all real values of x?

I. $\cos\left(x+\frac{\pi}{4}\right)=\cos x+\cos\frac{\pi}{4}$ II. $\sin(x+2\pi)=\sin x+\sin 2\pi$

III. $\sin(-x)=-\sin(x)$ IV. $\cos(5x)=5\cos x$

(**A**) I and II (**B**) I and III (**C**) $\boxed{\text{II and III}}$ (**D**) III and IV (**E**) I, II, III, and IV

43. Find the sum of the sequence of numbers $1, 2, 4, 5, 7, 8, 10, 11, ..., 176, 178, 179$.

(**A**) 10,740 (**B**) $\boxed{10{,}800}$ (**C**) 14,340 (**D**) 16,110 (**E**) None of these

44. In parallelogram $PQRS$, vertices P, Q, and R have coordinates $(-3,-2)$, $(1,5)$ and $(9,1)$, respectively. Find the coordinates of vertex S.

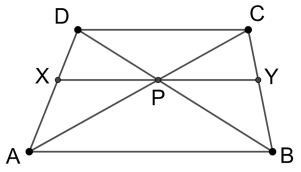
(**A**) $(4,-5)$ (**B**) $(5,4)$ (**C**) $\boxed{(5,-6)}$ (**D**) $(6,-5)$ (**E**) None of these

45. Find the number of solutions (x,y) to the system of equations

$$\begin{cases} 3y^2=xy \\ 2x^2+xy-84=0 \end{cases}$$

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) $\boxed{4}$ (**E**) None of these

46. In trapezoid $ABCD$, $CD=20$ and $AB=30$. Point P is the intersection of diagonals $\overline{BD}$ and $\overline{AC}$. Let X be the point on $\overline{AD}$ and Y be the point on $\overline{BC}$ so that $\overline{XY}$ goes through P and is parallel to $\overline{AB}$ and $\overline{CD}$. Find the length of $\overline{XY}$.



(**A**) $\boxed{24}$ (**B**) 25 (**C**) $16\sqrt{2}$ (**D**) $20\sqrt{2}$ (**E**) None of these

47. Find the sum of all real solutions to the equation $(x^2+7x+11)^{x^2+4x-12}=1$.

(**A**) -18 (**B**) $\boxed{-15}$ (**C**) -11 (**D**) -4 (**E**) None of these

48. For how many integer values of x will the rational expression $\dfrac{x^2+25}{x+5}$ simplify to an integer?

(**A**) 2 (**B**) 6 (**C**) $\boxed{12}$ (**D**) Infinitely many (**E**) None of these

49. Find the sum of the values A and B for which the following equation is true for all x:

$$24x+45=A(2x+3)+B(4x+5)$$

(**A**) 8 (**B**) $\boxed{21}$ (**C**) 45 (**D**) 69 (**E**) None of these

50. Find the height of a right pyramid with a square base whose area is 16 square units and whose lateral edges have length 5 units.

(**A**) 5 (**B**) $\boxed{\sqrt{17}}$ (**C**) $\sqrt{21}$ (**D**) $\sqrt{29}$ (**E**) None of these