The 37^{th} Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 24, 2018 at Regional Testing Centers
Second Round: April 14, 2018 at The University of Alabama at Birmingham

COMPREHENSIVE EXAM

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\widehat{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\widehat{AB}$ is the measure of $\widehat{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of North Alabama offers an undergraduate degree in Mathematics and has many great things to offer, including a new Mathematics Fellow program, an active undergraduate research group and a new Dual Degree Engineering program. For more information, go to www.una.edu/math.

1. The measures of the exterior angles of a hexagon are x, $2x$, $3x$, $3x$, $4x$, and $5x$. Find the measure of the largest interior angle.

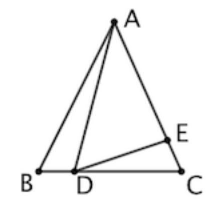
(**A**) $20°$ (**B**) $100°$ (**C**) $120°$ (**D**) $\boxed{160°}$ (**E**) None of these

2. There is exactly one integer a for which the polynomial $f(x) = ax^4 + 15x^3 - 5x^2 + 10x - a$ is divisible by $x + 3$. Find the sum of the value a and the coefficient on the x term of the quotient.

(**A**) $\boxed{10}$ (**B**) 16 (**C**) 20 (**D**) 24 (**E**) None of these

3. In the triangle shown, the measure of $\angle BAD = 38°$, $AB = AC$, and $AD = AE$. Find the measure of $\angle CDE$.

(**A**) $\boxed{19°}$ (**B**) $27°$ (**C**) $38°$ (**D**) $52°$ (**E**) None of these



4. Find the sum of the solutions to the equation $16^x - 7(4^x) = -10$.

(**A**) $\log_4 7$ (**B**) $\boxed{\log_4 10}$ (**C**) $\log_{16} 7$ (**D**) $\log_{16} 10$ (**E**) None of these

5. A number is *abundant* if the sum of its proper divisors is greater than the number itself. Recall that a proper divisor of a number n is any positive divisor which is less than n. Which of the following is an abundant number?

(**A**) 16 (**B**) $\boxed{20}$ (**C**) 22 (**D**) 28 (**E**) 32

6. Find the number of solutions (a, b), with a, b real numbers, to the system of equations

$$\begin{cases} y + |x| = 3 \\ |x|y + x^3 = 0 \end{cases}$$

(**A**) 1 (**B**) $\boxed{2}$ (**C**) 3 (**D**) 5 (**E**) None of these

7. A contest has ten entries. How many ways are there to choose first, second, and third place, along with two unordered honorable mentions?

(**A**) 252 (**B**) 2520 (**C**) $\boxed{15{,}120}$ (**D**) 30,240 (**E**) None of these

8. Find the absolute value of the sum of all solutions to the equation $(1 - 2x)(x + 6) = 18$.

(**A**) 3.5 (**B**) $\boxed{5.5}$ (**C**) 12 (**D**) 13.5 (**E**) None of these

9. Which of the following functions is one-to-one on its domain?

(**A**) $f(x) = x^3 - x$ (**B**) $f(x) = x^2 + 2$ (**C**) $f(x) = e^{x^2}$ (**D**) $\boxed{f(x) = \sqrt{x + 4}}$ (**E**) $f(x) = x - \dfrac{1}{x}$

10. How many integers are excluded from the solution set of the inequality $\dfrac{3x - 2}{x} > 1$?

(**A**) Zero (**B**) One (**C**) $\boxed{\text{Two}}$ (**D**) Infinitely Many (**E**) None of these

11. For how many real values of x is the equation $(x + 2)^3 = x^3 + 8$ true?

(**A**) Zero (**B**) One (**C**) $\boxed{\text{Two}}$ (**D**) Infinitely Many (**E**) None of these

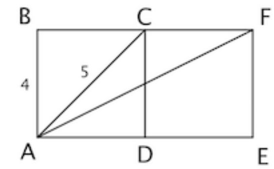
12. The rational expression $\dfrac{5x^2 - 2x + 3}{x^3 + 2x^2 + 5x + 10}$ is equivalent to the sum $\dfrac{A}{x+2} + \dfrac{Bx + C}{x^2 + 5}$. Find the product ABC.

(**A**) $\boxed{-36}$ (**B**) -495 (**C**) 0 (**D**) -30 (**E**) None of these

13. In the figure shown, $ABCD$ and $DCFE$ are rectangles, with $AB = 4$, $AC = 5$, and $BC = CF$. What is the perimeter of $\triangle ACF$?

(**A**) $\boxed{8 + 2\sqrt{13}}$ (**B**) $8 + 5\sqrt{2}$ (**C**) $9 + 4\sqrt{5}$

(**D**) $10 + 5\sqrt{2}$ (**E**) None of these



14. Find the maximum value of the function $f(x) = \dfrac{10}{4x^2 + 12x + 13}$.

(**A**) 2 (**B**) $\dfrac{1}{4}$ (**C**) $\dfrac{10}{13}$ (**D**) $\boxed{\dfrac{5}{2}}$ (**E**) None of these

15. The value of $\log 3.76$ to four decimal places is 0.5752. Find the value of $\log 37.6$.

(**A**) $\boxed{1.5752}$ (**B**) 3.627 (**C**) 5.752 (**D**) 10.5752 (**E**) None of these

16. Suppose for all positive integers n, we have $f(4 + n^2) = an + 2$ and $f(9 - n^2) = 3n - b$ for some numbers a and b. Then the value of $f(13)$ is

(**A**) $\boxed{-7}$ (**B**) -3 (**C**) 7 (**D**) not uniquely determined (**E**) not defined

17. Which of the following are true for all non-zero real values of x?

I. $\sin(3x) = 3\sin(x)$ II. $\dfrac{1}{x} + \dfrac{1}{2x} = \dfrac{1}{3x}$ III. $x^5(x^9) = x^{14}$

(**A**) I and II (**B**) I and III (**C**) I, II, and III (**D**) II only (**E**) $\boxed{\text{III only}}$

18. Find the sum of the solutions to the equation below where $0 \le x < 2\pi$.

$$\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} = 4$$

(**A**) π (**B**) $\boxed{2\pi}$ (**C**) 3π (**D**) 5π (**E**) None of these

19. If a boy were 3 months more than $3/5$ his current age, he would be 6.5 years old. How old is he currently?

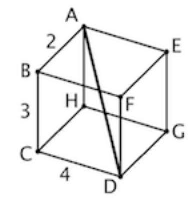
(**A**) 5 years, 10 months (**B**) 15 years, 10 months

(**C**) $\boxed{\text{10 years, 5 months}}$ (**D**) 11 years, 3 months (**E**) None of these

20. A cube has the same surface area as a rectangular solid whose dimensions are 16 inches long, 12 inches wide, and 9 inches high. Find the length of an edge of the cube, in inches.

(**A**) 12 (**B**) 24 (**C**) $\sqrt{39}$ (**D**) $\boxed{2\sqrt{37}}$ (**E**) None of these

21. A pole $\overline{AD}$ is to be placed in a rectangular box as shown, with dimensions of 2, 3, and 4 units. Find the length of the pole.



(**A**) $3\sqrt{3}$ (**B**) $\boxed{\sqrt{29}}$ (**C**) $\sqrt{34}$ (**D**) $\sqrt{41}$ (**E**) None of these

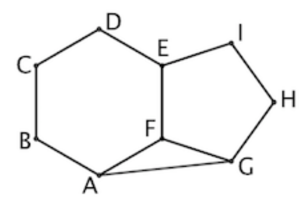
22. Let (x, y) be the Cartesian coordinates of the Polar point $(r, \theta) = \left(2, \dfrac{5\pi}{3}\right)$. Find $x + y$.

(**A**) $\boxed{1-\sqrt{3}}$ (**B**) $1+\sqrt{3}$ (**C**) $-1+\sqrt{3}$ (**D**) $-1-\sqrt{3}$ (**E**) None of these

23. Find the sum of the absolute values of all solutions to $(x-5)(x-7)(x+6)(x+4) = 504$.

(**A**) 2 (**B**) $\boxed{20}$ (**C**) 201 (**D**) 2018 (**E**) None of these

24. In the figure shown, $ABCDEF$ is a regular hexagon, while $EFGHI$ is a regular pentagon. Find the measure of angle $\angle AGF$.



(**A**) $\boxed{24^\circ}$ (**B**) 32° (**C**) 48° (**D**) 66° (**E**) None of these

25. If $\tan\theta = \dfrac{2}{5}$, find $\sin(2\theta)$.

(**A**) $\dfrac{4\sqrt{29}}{29}$ (**B**) $\dfrac{-4\sqrt{29}}{29}$ (**C**) $\boxed{\dfrac{20}{29}}$ (**D**) $-\dfrac{20}{29}$ (**E**) None of these

26. How many distinct arrangements are there of the letters of the word CALCULUS?

(**A**) 120 (**B**) $\boxed{5040}$ (**C**) 6720 (**D**) 40320 (**E**) None of these

27. What is the measure of an interior angle of an equiangular polygon with 54 diagonals?

(**A**) 108° (**B**) 120° (**C**) 135° (**D**) $\boxed{150^\circ}$ (**E**) None of these

28. Find the product of all solutions to the equation $x^2 + x = 2 - 2\sqrt{x^2 + x - 2}$.

(**A**) -8 (**B**) $\boxed{-2}$ (**C**) 0 (**D**) 12 (**E**) None of these

29. How many solutions (x, y) are there for the following system of equations?

$$\begin{cases} \log_y(x) = 4 \\ \log_y(9x) = 6 \end{cases}$$

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

30. The sum of the measures of the complement and supplement of an angle is 196°. Find the measure of the angle.

(**A**) $\boxed{37^\circ}$ (**B**) 40° (**C**) 42° (**D**) 52° (**E**) None of these

31. Suppose $f(x)$ is a degree six polynomial with positive leading coefficient, such that $f(-2) = -1$. What is the minimum number of real roots of the polynomial $f(x)$?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 4 (**E**) None of these

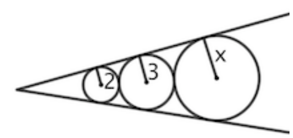
32. Find the area of the region enclosed by the graphs of the functions of $f(x) = |x|$ and $g(x) = 4$.

(**A**) 4 (**B**) 8 (**C**) $\boxed{16}$ (**D**) 32 (**E**) None of these

33. Find the largest integer value of n, with $0 \le n \le 100$, so that $(1 + i\sqrt{3})^n$ is a real number.

(**A**) 100 (**B**) $\boxed{99}$ (**C**) 98 (**D**) 97 (**E**) None of these

34. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the circle of radius 2 and both sides of the same angle. A third circle is tangent to the circle of radius 3 and both sides of the same angle. Find the radius of the third circle.



(**A**) 4 (**B**) $\boxed{4.5}$ (**C**) 5 (**D**) 5.5 (**E**) None of these

35. $LMNO$ is a square. P is a point inside the square such that LPO is an equilateral triangle. Determine the measure of $\angle PMN$.

(**A**) $\boxed{15^\circ}$ (**B**) 30° (**C**) 60° (**D**) 75° (**E**) None of these

36. If $g(x)$ is an even function, $h(x)$ is an odd function, and $f(x) = h(g(x)) - 2(h(x))^2$, what is true about the function $f(x)$?

(**A**) $\boxed{\text{Even}}$ (**B**) Odd (**C**) Neither even nor odd (**D**) Both even and odd (**E**) None of these

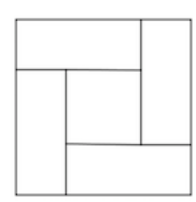
37. Let $g(x) = x^2 + 2x + 1$ and $f(x) = x^2 + 8x$. What is the value of the largest solution minus the smallest solution to the equation $(g \circ f)(x) = 9$?

(**A**) 2 (**B**) 11 (**C**) $4\sqrt{3}$ (**D**) $\boxed{6\sqrt{2}}$ (**E**) None of these

38. The equation $y^{-2} + 2y^{-1} - 15 = 0$ has two solutions a and b with $a > b$. Find $12a + 10b$.

(**A**) -14 (**B**) $-\dfrac{14}{15}$ (**C**) $\boxed{2}$ (**D**) 30 (**E**) None of these

39. A square is surrounded by four congruent rectangles. If each of the rectangles has perimeter of 22 units, determine the total sum of the areas of the four rectangles and the interior square.

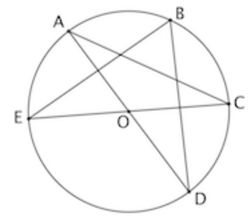


(**A**) 81 (**B**) $\boxed{121}$ (**C**) 225 (**D**) 484 (**E**) None of these

40. Given $f(x) = \dfrac{1}{x\sqrt{4x^2+9}}$, find an equivalent form for $f(x)$ in terms of trigonometric functions if $2x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(**A**) $\dfrac{2}{9}\cos\theta$ (**B**) $\dfrac{2}{9}\sin\theta$ (**C**) $\dfrac{1}{3}\cos\theta$ (**D**) $\boxed{\dfrac{2\cos^2\theta}{9\sin\theta}}$ (**E**) $\dfrac{2\sin^2\theta}{9\cos\theta}$

41. In the figure shown, points A, B, C, D and E all lie on the circle, point O is the center of the circle, and both $\overline{AD}$ and $\overline{CE}$ go through point O. Angle $\angle BEC$ has measure $28°$, and $\angle ADB$ has measure $37°$. Find the measure of $\angle ACE$.



(**A**) $14°$ (**B**) $20°$ (**C**) $\boxed{25°}$ (**D**) $37°$ (**E**) None of these

42. Let $\triangle ABC$ be a right triangle, with $m\angle ABC = 90°$, and $m\angle CAB = 30°$. Place a point D on $\overline{AB}$ 4 units from A such that $m\angle CDB = 60°$. What is the length of $\overline{BC}$?

(**A**) 2 (**B**) 6 (**C**) $\boxed{2\sqrt{3}}$ (**D**) $6\sqrt{3}$ (**E**) None of these

43. If $A = \begin{bmatrix} 2 & x \\ -1 & 4x \end{bmatrix}$, $B = \begin{bmatrix} y & 1 \\ 5 & 0 \end{bmatrix}$, and the product $AB = \begin{bmatrix} 27 & 2 \\ 54 & -1 \end{bmatrix}$, find the value $x + y$.

(**A**) 15.5 (**B**) $\boxed{9}$ (**C**) -3 (**D**) -13.5 (**E**) None of these

44. The difference quotient is defined as $\dfrac{f(x+h) - f(x)}{h}$ for $h \neq 0$. Evaluate and simplify the difference quotient for $f(x) = \sqrt{x}$.

(**A**) 1 (**B**) $\dfrac{1}{2\sqrt{x}}$ (**C**) $\dfrac{\sqrt{h}}{h}$ (**D**) $\dfrac{2x}{\sqrt{x+h} - \sqrt{x}}$ (**E**) $\boxed{\dfrac{1}{\sqrt{x+h} + \sqrt{x}}}$

45. Find the exact value of $\sin 6° \cos 9° + \cos 6° \sin 9°$.

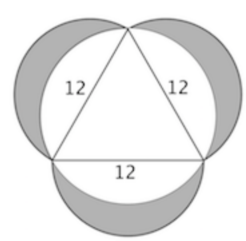
(**A**) $\boxed{\dfrac{\sqrt{6} - \sqrt{2}}{4}}$ (**B**) $\dfrac{\sqrt{2} + \sqrt{6}}{4}$ (**C**) $\dfrac{\sqrt{2} - \sqrt{6}}{4}$ (**D**) $\dfrac{\sqrt{3} + \sqrt{2}}{2}$ (**E**) $\dfrac{\sqrt{2} - \sqrt{3}}{2}$

46. Find the 12th term in the arithmetic sequence whose first three terms are

$$\frac{\sqrt{3}+1}{5}, \frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}-1}{5}$$

(**A**) $\boxed{\dfrac{12\sqrt{3} - 10}{5}}$ (**B**) $\dfrac{12\sqrt{3} - 1}{5}$ (**C**) $\dfrac{13\sqrt{3} - 11}{5}$ (**D**) $\dfrac{13\sqrt{3} + 13}{5}$ (**E**) None of these

47. In the figure shown, the equilateral triangle with a side length 12 is inscribed in a circle, and the three outer arcs are semicircles. Determine the shaded area in square units.



(**A**) $16\pi - 12\sqrt{3}$ (**B**) $56\pi + 12\sqrt{3}$ (**C**) $12\pi + 24\sqrt{3}$

(**D**) $\boxed{6\pi + 36\sqrt{3}}$ (**E**) None of these

48. Find the value of the continued fraction

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\ddots}}}}}$$

(A) $\boxed{\dfrac{\sqrt{15}-3}{2}}$ **(B)** $\dfrac{3}{2}$ **(C)** $\dfrac{1+\sqrt{29}}{14}$ **(D)** $\dfrac{-1+\sqrt{29}}{2}$ **(E)** None of these

49. Find the equation of the tangent line to the circle $(x-1)^2+(y+5)^2=25$ at the point $(4,-1)$.

(A) $y=-\frac{3}{4}x-5$ **(B)** $\boxed{y=-\frac{3}{4}x+2}$ **(C)** $y=\frac{5}{6}x-5$ **(D)** $y=\frac{5}{6}x-\frac{13}{3}$ **(E)** None of these

50. A right square pyramid with a base area of 16 and a height of 6 is cut halfway up parallel to the base. What is the surface area of the bottom half?

(A) $12+12\sqrt{10}$ **(B)** $\boxed{20+12\sqrt{10}}$ **(C)** $12+24\sqrt{2}$ **(D)** $20+24\sqrt{2}$ **(E)** None of these