The 35th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 27, 2016 at Regional Testing Centers
Second Round: April 9, 2016 at The University of North Alabama

COMPREHENSIVE EXAM

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees
- If A and B are points on a circle, then:
  $\overset{\frown}{AB}$ is the arc between A and B
  $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of North Alabama offers an undergraduate degree in Mathematics and has many great things to offer, including a new Mathematics Fellow program, an active undergraduate research group and a new Dual Degree Engineering program. For more information, go to www.una.edu/math.

1. Find the slope of the line between the points $(\sqrt{3}, \sqrt{6})$ and $(2, 4\sqrt{2})$.

(**A**) $6\sqrt{6} - 11\sqrt{2}$ (**B**) $\dfrac{5\sqrt{2} + 2\sqrt{6}}{26}$ (**C**) $5\sqrt{2}$ (**D**) $\boxed{5\sqrt{2} + 2\sqrt{6}}$ (**E**) None of these

2. Put the following values in order from least to greatest.

I. $\sin\left(\dfrac{5\pi}{4}\right)$ II. $\sec\left(\dfrac{3\pi}{4}\right)$ III. $\cos\left(-\dfrac{\pi}{3}\right)$ IV. $\tan\left(\dfrac{7\pi}{6}\right)$

(**A**) I, II, III, IV (**B**) $\boxed{\text{II, I, III, IV}}$ (**C**) IV, I, III, II (**D**) II, III, IV, II (**E**) None of these

3. In last year's NBA Finals, LeBron James scored a total of 169 points from 2-pointers and 3-pointers. If he made a total of 78 shots, how many 3-pointers did he make?

(**A**) 65 (**B**) 45 (**C**) 25 (**D**) 15 (**E**) $\boxed{\text{None of these(13)}}$

4. Given that the points $(1, 4)$, $(6, 12)$ and $(c, 10)$ are collinear, what is the value of c?

(**A**) 3.5 (**B**) 4 (**C**) 4.25 (**D**) $\boxed{4.75}$ (**E**) none

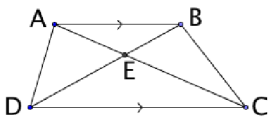
5. The equation $(x^2 - x - 4)^{3/4} - 2 = 6$ has two solutions, a and b. Find $a^2 + b^2$.

(**A**) 1 (**B**) 9 (**C**) 25 (**D**) $\boxed{41}$ (**E**) None of these

6. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the regular hexagon is $6\sqrt{3}$, then what is the area of the equilateral triangle?

(**A**) $\boxed{4\sqrt{3}}$ (**B**) $6\sqrt{3}$ (**C**) $8\sqrt{3}$ (**D**) $2\sqrt{3}$ (**E**) None of these

7. In a trapezoid $ABCD$ with $\overleftrightarrow{AB}$ parallel to $\overleftrightarrow{CD}$, the diagonals intersect at a point E. The area of triangle $\triangle ABE$ is 32 and of triangle $\triangle CDE$ is 50. Find the area of the trapezoid.



(**A**) 136 (**B**) $\boxed{162}$ (**C**) 178 (**D**) 184 (**E**) None of these

8. Find the absolute value of the sum of the two solutions to the equation $(3x - 2)(x + 3) = 14$.

(**A**) $\boxed{\frac{7}{3}}$ (**B**) $\frac{10}{3}$ (**C**) $\frac{11}{3}$ (**D**) $\frac{49}{3}$ (**E**) None of these

9. The College World Series for baseball is decided by a best of three game series; that is, the first team to win two games is the winner. If Texas and Florida are playing in the championship, and for any given game Texas has a 60% chance of beating Florida, what is the probability that the series will need to play the third game?

(**A**) 0.24 (**B**) 0.36 (**C**) $\boxed{0.48}$ (**D**) 0.52 (**E**) None of these

10. If $x + y = 3$ and $x^2 + y^2 = 6$, find $x^3 + y^3$.

(**A**) 7.5 (**B**) 9 (**C**) $\boxed{13.5}$ (**D**) 27 (**E**) None of these

11. Find the value of x so that $\log_4(2x + 1)^3 = 6$.

(**A**) $\dfrac{\sqrt[3]{6} - 1}{2}$ (**B**) $\dfrac{6\sqrt[3]{6} - 1}{2}$ (**C**) $\dfrac{1}{2}$ (**D**) $\boxed{\dfrac{15}{2}}$ (**E**) None of these

12. Two tangents to the same circle form a 50° angle. The radius of the circle is 10. Find the length of the smaller of the intercepted arcs.

(**A**) $\boxed{\frac{65\pi}{9}}$ (**B**) $\frac{40\pi}{9}$ (**C**) $\frac{25\pi}{9}$ (**D**) $\frac{10\pi}{9}$ (**E**) None of these

13. On a date night menu, you and your date get to choose one appetizer to share from a list of three, two entrees from a list of five, and one dessert to share from a list of four. Assuming you both could order the same entree, how many different meals are possible?

(**A**) 120 (**B**) 160 (**C**) 240 (**D**) $\boxed{300}$ (**E**) None of these

14. The system of equations listed below has two solutions (a, b) and (c, d), where $a < c$. Find the number $a - b + 2c + 3d$.

$$\begin{cases} \frac{1}{x} + \frac{1}{y} &= \frac{1}{2} \\ \frac{2}{xy} &= \frac{1}{9} \end{cases}$$

(**A**) -27 (**B**) $\boxed{18}$ (**C**) 27 (**D**) 30 (**E**) None of these

15. Define the function $f(x-2) = (x-2)^2 - 3(x-3) + 1$. Find $f(-1)$.

(**A**) -2 (**B**) 5 (**C**) $\boxed{8}$ (**D**) 22 (**E**) None of these

16. The area of a square inscribed in a circle is 30 units. Determine the perimeter of the square inscribed in a semicircle of the same radius.

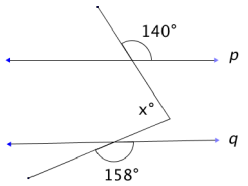
(**A**) $4\sqrt{15}$ (**B**) $4\sqrt{30}$ (**C**) $\boxed{8\sqrt{3}}$ (**D**) $8\sqrt{15}$ (**E**) None of these

17. Find the sum of all solutions to the equation $|x+3|^2 + 7|x+3| - 18 = 0$.

(**A**) $\boxed{-6}$ (**B**) -7 (**C**) -12 (**D**) 7 (**E**) None of these

18. Assume $p \parallel q$ in the figure shown. Find the angle supplementary to angle x.

(**A**) 112° (**B**) 122° (**C**) 128° (**D**) 138° (**E**) $\boxed{\text{None of these } (118^\circ)}$



19. Find the sum of all solutions to the equation $\cos(2x) = -\sin(2x)$ for x in $[0, 2\pi]$.

(**A**) $\frac{5\pi}{4}$ (**B**) $\frac{5\pi}{2}$ (**C**) $\boxed{\frac{9\pi}{2}}$ (**D**) 9π (**E**) None of these

20. Define two new operations on real numbers as follows

$$a \bowtie b = a^b \qquad a \circledast b = 2a + b$$

Calculate the value of $(16 \bowtie \frac{1}{2}) \circledast (2 \bowtie 3)$.

(**A**) 14 (**B**) $\boxed{16}$ (**C**) 22 (**D**) 24 (**E**) None of these

21. Find the value of $i \cdot i^2 \cdots i^{63}$, where $i = \sqrt{-1}$.

(**A**) $\boxed{1}$ (**B**) -1 (**C**) i (**D**) $-i$ (**E**) None of these

22. Given that it is 3 o'clock, exactly how long will it take for the minute hand to catch up with the hour hand?

(**A**) 15 min (**B**) $\boxed{16\ \frac{4}{11}\ \text{min}}$ (**C**) $16\ \frac{1}{2}$ min (**D**) 17 min (**E**) None of these

23. If $f^{-1}(x) = \sqrt{x-3}+5$, is the inverse of the function $f(x)$, find the solution to the equation $f(x) = 52$.

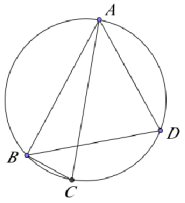
(**A**) -2 (**B**) $\boxed{12}$ (**C**) 212 (**D**) 2212 (**E**) None of these

24. A parallelogram's vertices have coordinates of $(0,0)$, $(1,\sqrt{3})$, $(4,0)$ and $(5,\sqrt{3})$. What is the area of the parallelogram?

(**A**) $2\sqrt{3}$ (**B**) 4 (**C**) $\boxed{4\sqrt{3}}$ (**D**) 8 (**E**) None of these

25. Points A, B, C and D lie on a circle with $\overline{AC}$ a diameter, $AB = 4$ and $BC = 2$, $\angle ABD \cong \angle CBD$. What is BD?

(**A**) $2\sqrt{3}$ (**B**) $5\sqrt{2}$ (**C**) $3\sqrt{3}$ (**D**) $\boxed{3\sqrt{2}}$ (**E**) None of these



26. Find the sum of all values of x so that the matrix $A = 3\begin{bmatrix} 2-x & x+3 \\ -1 & 2x \end{bmatrix}$ has a determinant of 0.

(**A**) $\frac{15}{2}$ (**B**) $\boxed{\frac{5}{2}}$ (**C**) $-\frac{5}{6}$ (**D**) $-\frac{5}{2}$ (**E**) None of these

27. For what value of k are the lines $2x + 3y = 4k$ and $x - 2ky = 7$ perpendicular?

(**A**) $-\frac{3}{4}$ (**B**) $\frac{1}{6}$ (**C**) $\boxed{\frac{1}{3}}$ (**D**) $\frac{1}{2}$ (**E**) None of these

28. For which of the following values of k does the equation $\dfrac{x-1}{x-2} = \dfrac{x-k}{x-6}$ have no solution for x?

(**A**) 1 (**B**) 3 (**C**) $\boxed{5}$ (**D**) 7 (**E**) None of these

29. Find the equation of the line which is the perpendicular bisector of the line segment connecting the points $(1,4)$ and $(-3,6)$.

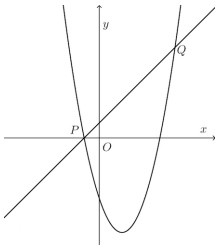
(**A**) $\boxed{y = 2x + 7}$ (**B**) $y = \frac{1}{2}x + \frac{11}{2}$ (**C**) $y = -2x + 3$ (**D**) $y = -\frac{1}{2}x + \frac{9}{2}$ (**E**) None of these

30. Find the exact value of

$$\frac{(10!)^2 - (9!)^2}{(10!)^2 + (9!)^2}.$$

(**A**) $\frac{9}{11}$ (**B**) $\boxed{\frac{99}{101}}$ (**C**) $\frac{81}{121}$ (**D**) $\frac{19}{181}$ (**E**) None of these

31. In the figure shown, the line $y = x + 1$ intersects the parabola $y = x^2 - 3x - 4$ at points P and Q. What are the coordinates of point Q?



(**A**) $(-1, 0)$ (**B**) $(4, 0)$ (**C**) $(4, 5)$ (**D**) $\boxed{(5, 6)}$ (**E**) None of these

32. Find the number of distinct solutions to the equation $\sqrt[3]{5x} = \sqrt{2x}$.

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) None of these

33. A box contains four fair coins and six biased coins. Whenever a fair coin is flipped, it lands with heads facing up with a probability of 0.5. Whenever a biased coin is flipped it lands with heads facing up with a probability of 0.9. A coin is randomly chosen from the box and flipped. What is the probability that it will land with heads facing up?

(**A**) $\boxed{0.74}$ (**B**) 0.70 (**C**) 0.50 (**D**) 0.45 (**E**) None of these

34. The smallest angle of a rhombus is one-half the measure of the larger angle. The shorter diagonal is 20 m. Find the perimeter of the rhombus.

(**A**) 40 m (**B**) 60 m (**C**) $\boxed{80 \text{ m}}$ (**D**) 100 m (**E**) None of these

35. Find the sum of all solutions to the equation $e^{2x} - 5e^x = -6$.

(**A**) 5 (**B**) $\ln 5$ (**C**) 6 (**D**) $\boxed{\ln 6}$ (**E**) None of these

36. How many solutions are there to the equation $\cos(4x) = 2$ for x in $[0, 2\pi]$?

(**A**) 1 (**B**) 2 (**C**) 4 (**D**) 8 (**E**) $\boxed{\text{None of these (0)}}$

37. Clara is on a bike ride. If she bikes uphill 4 miles at a rate of 8 miles per hour and then bikes downhill 4 miles at a rate of 16 miles per hour, what was her average speed on the trip, rounded to the nearest integer?

(**A**) 10 (**B**) $\boxed{11}$ (**C**) 12 (**D**) 13 (**E**) None of these

38. Find the sum of the squares of all values of x so that $f(x) = 8$ if

$$f(x) = \begin{cases} x^2 + 2x & x < 0 \\ 3x - 7 & x \geq 0 \end{cases}$$

(**A**) 20 (**B**) 25 (**C**) $\boxed{41}$ (**D**) 45 (**E**) None of these

39. If $f(x) = a + bx$, there are two such pairs (a, b) such that $f(f(1)) = \frac{23}{2}$ and $f(f(0)) = \frac{5}{2}$. What is the maximum value of $\frac{a}{b}$?

(**A**) $\frac{5}{24}$ (**B**) $\frac{5}{18}$ (**C**) $\boxed{\frac{5}{12}}$ (**D**) $\frac{5}{6}$ (**E**) None of these

40. Suppose that $f(x) = x^x$ and $g(x) = x^{2x}$. Which of the functions below is equivalent to $(f \circ g)(x)$?

(**A**) x^{3x} (**B**) $x^{x^{2x}}$ (**C**) $x^{2x^{x+1}}$ (**D**) $\boxed{x^{2x^{2x+1}}}$ (**E**) $(2x)^{2x}$

41. Quadrilateral $ABCD$ is inscribed in a circle with $m\angle C = 85°$. Find $m\angle A$.

(**A**) $85°$ (**B**) $\boxed{95°}$ (**C**) $105°$ (**D**) $115°$ (**E**) Not enough information

42. Find the number of distinguishable permutations of the letters of the word PARALLEL.

(**A**) 120 (**B**) $\boxed{3360}$ (**C**) 5040 (**D**) 40320 (**E**) None of these

43. The difference quotient of a function $f(x)$ is the quotient $\dfrac{f(x+h)-f(x)}{h}$, $h \neq 0$. Find the difference quotient of x^2+1.

(**A**) 1 (**B**) h (**C**) $\dfrac{h^2+1}{h}$ (**D**) $2x$ (**E**) $\boxed{2x+h}$

44. Find the value of c so that $x = 1$ is not a vertical asymptote of the graph of $y = \dfrac{x^2-4x+c}{x^2+x-2}$.

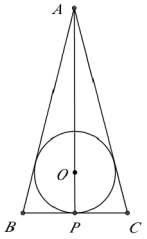
(**A**) -12 (**B**) -5 (**C**) $\boxed{3}$ (**D**) 4 (**E**) None of these

45. Which of the following classical constructions is impossible using only a compass and a straight edge?

I. Draw the perpendicular bisector of a segment
II. Find the midpoint of a segment
III. Inscribe a regular hexagon in a circle
IV. Trisect an angle

(**A**) $\boxed{\text{IV Only}}$ (**B**) III and IV (**C**) I and III (**D**) II Only (**E**) I and II

46. In the figure shown, the circle is inscribed in an isosceles triangle $\triangle ABC$, with segment $\overline{AP}$ passing through center O of the circle, with $AC = AB = 12$ and $BP = 4$. Find the radius of the circle.



(**A**) $2\sqrt{3}$ (**B**) $4\sqrt{2}$ (**C**) $4\sqrt{3}$ (**D**) $\boxed{2\sqrt{2}}$ (**E**) None of these

47. The function $p(x) = -x^2 + 46x - 360$ models the daily profits, in hundreds of dollars, for a small company that produces x bicycles per day. How many bicycles should be made each day to maximize profit?

(**A**) $\boxed{23}$ (**B**) 46 (**C**) 169 (**D**) 180 (**E**) None of these

48. Which of the following is equivalent to $\sin^2(x)\cos^2(x)$?

(**A**) $\boxed{\frac{1}{8} - \frac{1}{8}\cos(4x)}$ (**B**) 1 (**C**) $\frac{1}{2}\cos(4x) - \frac{1}{4}$ (**D**) $\frac{1}{8} - \frac{1}{2}\cos(x)$ (**E**) 0

49. Find the area of the region of the plane satisfying $1 \leq x^2 + y^2 \leq 9$ and $y \leq |x|$.

(**A**) 2π (**B**) 4π (**C**) $\boxed{6\pi}$ (**D**) 8π (**E**) None of these

50. Form a 7-digit number, n, at random using the digits 1 through 7 exactly once. What is the probability that n is divisible by 25?

(**A**) $\frac{1}{14}$ (**B**) $\boxed{\frac{1}{21}}$ (**C**) $\frac{1}{42}$ (**D**) $\frac{1}{2520}$ (**E**) None of these