The 39th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 29, 2020 at Regional Testing Centers
Second Round: April 25, 2020 at Auburn University at Montgomery

COMPREHENSIVE EXAMINATION

Construction of this test directed
by
Ashley Johnson, University of North Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

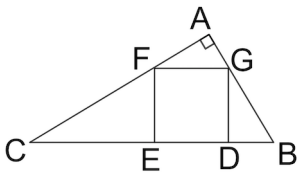
- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Evaluate $\sin^2\left(\frac{\pi}{3}\right)\cos(\pi) + \cot\left(\frac{\pi}{4}\right)$.

(**A**) $\boxed{\frac{1}{4}}$ (**B**) $\frac{3}{4}$ (**C**) 1 (**D**) $\frac{7}{4}$ (**E**) None of these

2. The square $DEFG$ has its vertices on the three sides of right triangle $\triangle ABC$ as shown in the figure. If the area of $DEFG$ is 36, and $BD = 3$, find the area of $\triangle ABC$.



(**A**) $\frac{63\sqrt{5}}{5}$ (**B**) $\frac{108\sqrt{5}}{5}$ (**C**) $\frac{324}{5}$ (**D**) $\boxed{\frac{441}{5}}$ (**E**) None of these

3. The equation $x^4 = y^2 + 71$ has only one solution (a, b) where both a, b are positive integers. Find $a + b$.

(**A**) 37 (**B**) $\boxed{41}$ (**C**) 48 (**D**) 55 (**E**) None of these

4. Find the equation of the line that is the perpendicular bisector of the line segment with end points $(8, 14)$ and $(2, 6)$.

(**A**) $3x - 4y = -35$ (**B**) $4x - 3y = -10$

(**C**) $3x + 4y = 50$ (**D**) $\boxed{3x + 4y = 55}$ (**E**) None of these

5. Find the number of solutions to the equation $2^{-|x|} = x^2 + 1$.

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

6. Three vertices of parallelogram $PQRS$ are $P(-3, -2)$, $Q(1, -5)$ and $R(9, 1)$, with P and R diagonally opposite. What is the sum of the coordinates of vertex S?

(**A**) 7 (**B**) 8 (**C**) $\boxed{9}$ (**D**) 10 (**E**) None of these

7. If $f(x) = \dfrac{x}{3x+1}$ and $(f \circ g)(x) = x$, find $g(x)$.

(**A**) $\boxed{g(x) = \dfrac{x}{1-3x}}$ (**B**) $g(x) = \dfrac{3x+1}{x}$

(**C**) $g(x) = \dfrac{1}{3x+1}$ (**D**) $g(x) = \dfrac{1}{1-3x}$ (**E**) $g(x) = \dfrac{1-3x}{x}$

8. Find the sum of the squares of all solutions to the equation $(x - 2)(6x + 2) = 20$.

(**A**) 8 (**B**) 493 (**C**) $\frac{25}{9}$ (**D**) $\boxed{\frac{97}{9}}$ (**E**) None of these

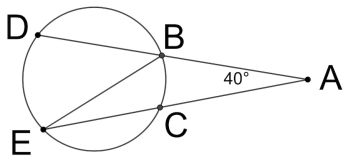
9. Let $f(x) = 2x^2 + x + 2$ and $g(x) = \frac{1}{2}x$. What is the minimum vertical distance between the graphs of $f(x)$ and $g(x)$?

(**A**) $\frac{61}{32}$ (**B**) $\boxed{\frac{63}{32}}$ (**C**) 2 (**D**) $\frac{67}{32}$ (**E**) None of these

10. In triangle $\triangle ABC$, $AB = 16$, $AC = 9$ and the median from vertex C has length 11. Find the perimeter of $\triangle ABC$.

(**A**) 36 (**B**) $\boxed{42}$ (**C**) 50 (**D**) 53 (**E**) None of these

11. In the figure shown, points D, B, E and C are on the circle, A, B, and D are colinear, A, C, and E are colinear, $m\angle DAE = 40°$, and $m\overset{\frown}{BD} = m\overset{\frown}{DE} = m\overset{\frown}{CE}$. Determine the measure of $\angle DBE$.



(**A**) $\boxed{55°}$ (**B**) 60° (**C**) 65° (**D**) 70° (**E**) None of these

12. A survey of 140 employees at an accounting firm yielded the following information:
62 were under age 40
48 were 40 or older and making \$50,000 or more
40% of the employees making less than \$50,000 were under age 40
Find the probability that a randomly selected employee is under age 40 and making \$50,000 or more.

(**A**) $\boxed{\frac{3}{10}}$ (**B**) $\frac{26}{35}$ (**C**) $\frac{11}{14}$ (**D**) $\frac{17}{140}$ (**E**) None of these

13. Find the x-coordinate of the point on the x-axis that is equidistant to $(1, -1)$ and $(-5, 5)$.

(**A**) $\boxed{-4}$ (**B**) -3 (**C**) -2 (**D**) 0 (**E**) None of these

14. How many positive roots does the polynomial $f(x) = 3x^6 - 4x^4 - x - 2$ have?

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

15. Find the distance between the foci in the ellipse given by the equation $x^2 + 9y^2 - 4x - 72y + 139 = 0$.

(**A**) $\boxed{4\sqrt{2}}$ (**B**) $2\sqrt{10}$ (**C**) 16 (**D**) 20 (**E**) None of these

16. What is the remainder when $x^{87} + x^{42} - x^{31} + x^5 + 1$ is divided by $x^3 + x$?

(**A**) $x + 1$ (**B**) $x^2 + x$ (**C**) $x^2 + 1$ (**D**) $\boxed{x^2 + x + 1}$ (**E**) None of these

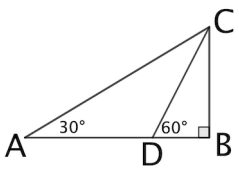
17. How many numbers of the form $4a37b$, where both a and b represent single digit numbers, are divisible by 36?

(**A**) $\boxed{2}$ (**B**) 4 (**C**) 8 (**D**) 16 (**E**) None of these

18. Find the value of $\log_2(81)\log_9(32)$.

(**A**) 7 (**B**) 8 (**C**) 9 (**D**) $\boxed{10}$ (**E**) None of these

19. In right triangle $\triangle ABC$, the point D on $\overline{AB}$ has $AD = 8$, $m\angle CDB = 60°$, and $m\angle CAB = 30°$. Find BC.



(**A**) $\boxed{4\sqrt{3}}$ (**B**) $4\sqrt{5}$ (**C**) $6\sqrt{3}$ (**D**) $6\sqrt{5}$ (**E**) None of these

20. The following system has a unique solution (a, b) satisfying it. Find ab.

$$\begin{cases} 2^{x+1} - 3^{y+2} = 15 \\ 2^x + 3^{y+1} = \frac{25}{3} \end{cases}$$

(**A**) $\frac{4}{27}$ (**B**) $\frac{8}{9}$ (**C**) $\frac{16}{9}$ (**D**) $\frac{32}{27}$ (**E**) $\boxed{\text{None of these}}$

21. The interior angle measures in degrees of a pentagon are $2x + 30$, $2x + 50$, $2x + 70$, $2x + 90$, and $2x + 110$. Find the measure of the largest angle, in degrees.

(**A**) 108 (**B**) 132 (**C**) 140 (**D**) $\boxed{148}$ (**E**) None of these

22. Find the smallest positive value of x for which the graphs of $y = 1 + \cos 2x$ and $y = 2 - \sin x$ intersect.

(**A**) 0 (**B**) $\boxed{\frac{\pi}{6}}$ (**C**) $\frac{\pi}{3}$ (**D**) π (**E**) None of these

23. For what range of values of k do the lines $kx + y = 3$ and $x - y = 2$ intersect in the first quadrant?

(**A**) $k < \frac{1}{2}$ (**B**) $2 < k$ (**C**) $\boxed{-1 < k < \frac{3}{2}}$ (**D**) $2 < k < 3$ (**E**) None of these

24. An unfair coin which comes up heads with a probability of 0.80 is flipped 10 times. What is the probability it comes up heads exactly once?

(**A**) $\frac{1}{10}$ (**B**) $\frac{4}{5}$ (**C**) $\frac{4}{5^{10}}$ (**D**) $\boxed{\frac{8}{5^9}}$ (**E**) None of these

25. Find the sum of all solutions on the interval $[0, 2\pi)$ to the equation $\cos^2 x + 2\sin^2 x = \frac{5}{4}$.

(**A**) $\frac{\pi}{2}$ (**B**) π (**C**) 2π (**D**) $\boxed{4\pi}$ (**E**) None of these

26. The three-digit numbers acb, $a79$, $b0c$, and $bb1$ are consecutive terms in an arithmetic sequence, where a, b, and c each represent a single digit. How many possible values are there for a?

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

27. Which of the following (x, y) pairs is a solution to the equation $x^4 + 2x^2y^2 + 9y^4 = 0$?

(**A**) $(-179 + 179\sqrt{3}\,i, 179)$ (**B**) $(179, -179 + 179\sqrt{2}\,i)$ (**C**) $(179 - 179\sqrt{3}\,i, 179)$

(**D**) $(179, -179 + 179\sqrt{3}\,i)$ (**E**) $\boxed{(-179 + 179\sqrt{2}\,i, 179)}$

28. Find the exact value of $\sec\left(\sin^{-1}\left(\frac{1}{4}\right)\right)$.

(**A**) $\boxed{\frac{4}{\sqrt{15}}}$ (**B**) $\frac{1}{\sqrt{3}}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{4}{\sqrt{17}}$ (**E**) None of these

29. The difference quotient of a function $f(x)$ is defined as $\dfrac{f(x+h) - f(x)}{h}$ for $h \neq 0$. Evaluate and simplify the difference quotient for $f(x) = x^3 + 6x$.

(**A**) $3x^2 + xh + h^2 + 6$ (**B**) 1 (**C**) $\boxed{3x^2 + 3xh + h^2 + 6}$ (**D**) $h^2 + 6$ (**E**) $3x^2 + 6xh + 6$

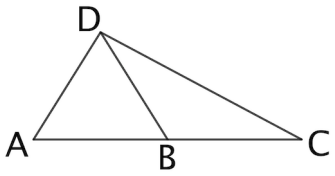
30. The ceiling of a number x, denoted as $\lceil x \rceil$, is the smallest integer that is greater than or equal to x. The floor of a number x, denoted as $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. For example, $\lceil 1.1 \rceil = 2$ and $\lfloor 1.1 \rfloor = 1$. Find $\lceil -2.4 \rceil + \lfloor 8.9 \rfloor$.

(**A**) 5 (**B**) $\boxed{6}$ (**C**) 7 (**D**) 8 (**E**) None of these

31. A right rectangular prism has surface area of 1000 square inches, and has a base width and length of 10 inches and 20 inches, respectively. Find the volume of the prism in cubic inches.

(**A**) 1500 (**B**) $\boxed{2000}$ (**C**) 3000 (**D**) 4000 (**E**) None of these

32. In the figure, point B is the midpoint of $\overline{AC}$, and point D is placed so that $DA = DB$, and $DB = BC = 10$. Find DC.



(**A**) $10\sqrt{2}$ (**B**) $\boxed{10\sqrt{3}}$ (**C**) $20\sqrt{2}$

(**D**) $20\sqrt{3}$ (**E**) None of these

33. Find the distance between the two points of intersection of the circle given by $(x-3)^2 + (y+5)^2 = 9$ and the line $x + y = 1$.

(**A**) $\boxed{3\sqrt{2}}$ (**B**) $4\sqrt{2}$ (**C**) $3\sqrt{6}$ (**D**) 6 (**E**) None of these

34. Let $f(x) = \log_2(x)$. Which of the following is equal to $f(x^2) - f(4x) + f(8)$?

(**A**) $f(x) - 5$ (**B**) $3f(x) + 1$ (**C**) $[f(x)]^2 - 4f(x) + 3$ (**D**) $\boxed{f(x) + 1}$ (**E**) $f(2x - 4)$

35. Let $D = a^2 + b^2 + (ab)^2$, where a, b are consecutive integers. Then $\sqrt{D}$ is:

(**A**) always an even integer (**B**) sometimes an odd integer, sometimes not

(**C**) $\boxed{\text{always an odd integer}}$ (**D**) sometimes rational, sometimes not (**E**) always irrational

36. The solutions of the equation $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. What is the median of the solutions?

(**A**) $\frac{4}{15}$ (**B**) $\frac{4}{5}$ (**C**) $\frac{3}{8}$ (**D**) $\boxed{\frac{3}{4}}$ (**E**) None of these

37. Define an operation $*$ on ordered pairs of integers as $(a, b) * (c, d) = (a + c, b^2 - d^2)$. Find $(-2, 4) * (0, 1)$.

(**A**) $\boxed{(-2, 15)}$ (**B**) $(-2, 9)$ (**C**) $(2, 15)$ (**D**) $(2, 9)$ (**E**) None of these

38. The number $x = \sqrt[3]{9 + \sqrt{17}} + \sqrt[3]{9 - \sqrt{17}}$ is a root of which of the following polynomials?

(**A**) $x^3 + 12x + 18$ (**B**) $x^3 - 12x + 18$ (**C**) $\boxed{x^3 - 12x - 18}$ (**D**) $x^3 - 18$ (**E**) $x^3 - 12x$

39. Triangle $\triangle ABC$ is isosceles with base $\overline{AC}$. Points P and Q are on $\overline{BC}$ and $\overline{AB}$, respectively, so that $AC = AP = PQ = QB$. Find the measure of angle $\angle ABC$, in degrees.

(**A**) 18 (**B**) 20 (**C**) $\frac{45}{2}$ (**D**) $\boxed{\frac{180}{7}}$ (**E**) None of these

40. Find the sum of all real solutions to the equation $28 \cdot 2^x + 5 \cdot 4^x - 8^x = 32$

(**A**) 1 (**B**) $\boxed{3}$ (**C**) 5 (**D**) 9 (**E**) None of these

41. A man planning to walk 40 miles realizes that by walking one mile per hour faster, he could make the journey in 2 hours less time. How many miles per hour was he originally going to be walking?

(**A**) 3 (**B**) 3.5 (**C**) $\boxed{4}$ (**D**) 4.5 (**E**) None of these

42. A rectangle is three times as long as it is wide. If it has a diagonal of length 10, what is the area of the rectangle?

(**A**) 10 (**B**) 20 (**C**) $\boxed{30}$ (**D**) 40 (**E**) None of these

43. If $f(x) = ax^2 + bx + c$ is a quadratic function with vertex at $(2, 4)$, going through point $(1, 3)$, what is $a + b + c$?

(**A**) $\boxed{3}$ (**B**) 4 (**C**) 6 (**D**) 7 (**E**) None of these

44. Which of the following have the same value as $\sin 38°$?

(**A**) $\cos 38°$ (**B**) $\sin 52°$ (**C**) $\cos 142°$ (**D**) $\sin 218°$ (**E**) $\boxed{\cos 308°}$

45. The graph of a function $g(x)$ is symmetric about the y-axis. What must be true about $g(x)$?

(**A**) $\boxed{g(x) \text{ is an even function}}$ (**B**) $g(x)$ is an odd function

(**C**) $g(x)$ has an even number of roots (**D**) $g(x)$ has an odd number of roots (**E**) None of these

46. In triangle ABC, $AB = 3$, $BC = 4$ and $AC = 6$. If $\overline{BC}$ is extended past C to point D so that $CD = BC$, find AD.

(**A**) $2\sqrt{13}$ (**B**) $3\sqrt{17}$ (**C**) $\sqrt{91}$ (**D**) $\boxed{\sqrt{95}}$ (**E**) None of these

47. Triangle $\triangle ABC$ is formed by the vertices $A(8, 3)$, $B(4, 1)$ and $C(-5, 4)$. Determine the length of the altitude to side $\overline{AB}$.

(**A**) $\boxed{3\sqrt{5}}$ (**B**) $5\sqrt{5}$ (**C**) $7\sqrt{5}$ (**D**) $9\sqrt{5}$ (**E**) None of these

48. How many different arrangements are there of the letters in the word PUPPIES?

(**A**) 35 (**B**) 120 (**C**) $\boxed{840}$ (**D**) 5040 (**E**) None of these

49. Find the value of x in base 5 satisfying the base 5 equation given by $3_5x + 2_5 = 43_5$.

(**A**) 7_5 (**B**) $\boxed{12_5}$ (**C**) 15_5 (**D**) 30_5 (**E**) None of these

50. Find the y-coordinate of the y-intercept of a line with an x-intercept of $(3, 0)$, and a slope of $-\frac{5}{7}$.

(**A**) $-\dfrac{7}{15}$ (**B**) $\dfrac{7}{15}$ (**C**) $-\dfrac{15}{7}$ (**D**) $\boxed{\dfrac{15}{7}}$ (**E**) None of these