The 34th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 28, 2015 at Regional Testing Centers
Second Round: April 11, 2015 at University of North Alabama

COMPREHENSIVE EXAM

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees
- If A and B are points on a circle, then:
  $\overset{\frown}{AB}$ is the arc between A and B
  $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

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Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

The following information is courtesy of the U.S. Bureau of Labor Statistics.

- The median salary of a Mathematician in 2012 was $101,360 per year.
- Over the next 10 years, the job opportunities for mathematicians are expected to grow by 23%!

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of North Alabama offers an undergraduate degree in Mathematics and has many great things to offer, including a new Mathematics Fellow program, an active undergraduate research group and a new Dual Degree Engineering program. For more information, go to www.una.edu/math.

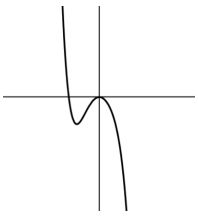
1. For $f(x) = 6x - 4(2x-1)^2 + 5$, find $f(2)$.

(**A**) $\boxed{-19}$ (**B**) -29 (**C**) 77 (**D**) -127 (**E**) None of these

2. The area of a particular equilateral triangle is equal to its perimeter. What is the length of a side of this triangle?

(**A**) $\sqrt{3}$ (**B**) $2\sqrt{3}$ (**C**) $\boxed{4\sqrt{3}}$ (**D**) $6\sqrt{3}$ (**E**) None of these.

3. Which of the following could be the leading term of the polynomial whose graph is as pictured?



(**A**) $3x^3$ (**B**) $-x^4$ (**C**) $\boxed{-2x^5}$ (**D**) $8x^6$ (**E**) None of these

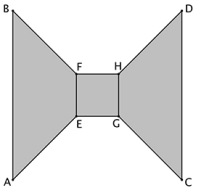
4. Exactly how many solutions are there to the equation $|x-2| + |x-3| = 1$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) $\boxed{\text{None of these}}$

5. The difference quotient of a function $f(x)$ is the quotient $\dfrac{f(x+h)-f(x)}{h}, h \neq 0$. Find the difference quotient of $f(x) = \frac{3}{x}$.

(**A**) $\boxed{\frac{-3}{x(x+h)}}$ (**B**) 1 (**C**) $\frac{1}{x(x+h)}$ (**D**) $\frac{3}{x(x+h)}$ (**E**) $\frac{3}{h^2}$

6. In the figure to the right, $m\angle A = m\angle B = m\angle C = m\angle D = 45°$, $AE = BF = HD = CG$, $AB = CD$, and $EFHG$ is a square. How many lines of reflection symmetry does the figure have?



(**A**) 1 (**B**) $\boxed{2}$ (**C**) 3 (**D**) 4 (**E**) None of these.

7. If the shadow of a building increases by 10 meters when the angle of elevation of the sun's rays measured from the ground decreases from 60° to 45°, what is the height of the building?

(**A**) $\frac{25}{2} + \frac{5\sqrt{3}}{2}$ (**B**) $20 + 5\sqrt{2}$ (**C**) $\boxed{15 + 5\sqrt{3}}$ (**D**) $25 + 5\sqrt{3}$ (**E**) None of these

8. Find the smallest real solution to the equation $\ln 15 - \ln(x-2) = \ln x$.

(**A**) $\frac{13}{2}$ (**B**) $\boxed{5}$ (**C**) $\frac{17}{2}$ (**D**) 3 (**E**) None of these

9. A square $ABCD$ of side length 1 has a point E on $\overline{CD}$ and point F in the interior of the square so that $\overline{EF} \perp \overline{DC}$ and $\overline{AF} \cong \overline{BF} \cong \overline{EF}$. Determine the area of $AFBCD$.

(**A**) $\frac{5}{8}$ (**B**) $\frac{9}{16}$ (**C**) $\frac{7}{8}$ (**D**) $\boxed{\frac{13}{16}}$ (**E**) None of these.

10. The graphs of equations $9x^2 + y^2 = 9$ and $3x + 2y = 6$ intersect at points (a, b) and (c, d). Find $a + b + c + d$.

(**A**) $\boxed{\frac{28}{5}}$ (**B**) $-\frac{13}{8}$ (**C**) 4 (**D**) 0 (**E**) None of these

11. In triangle $\triangle ABC$, the interior angle at vertex B has measure $72°$, and the exterior angle at vertex A has measure $145°$. Find the measure of the exterior angle at vertex C.

(**A**) $73°$ (**B**) $83°$ (**C**) $\boxed{107°}$ (**D**) $135°$ (**E**) None of these.

12. Which of the following functions is/are equal to $f(x) = 4^x$?

I. $g(x) = 2^{2^x}$ II. $h(x) = (2^x)^2$ III. $k(x) = 2^{2x}$

(**A**) I only (**B**) I and III only (**C**) I, II and III (**D**) II only (**E**) $\boxed{\text{II and III only}}$

13. If $x + \dfrac{1}{x} = 3$, then the value of $x^3 + \dfrac{1}{x^3}$ is

(**A**) 9 (**B**) $\boxed{18}$ (**C**) 24 (**D**) 27 (**E**) None of these

14. A chord of length 24 cm is drawn in a circle of radius 13 cm. Find the distance from the chord to the center of the circle.

(**A**) 4.5 cm (**B**) $\boxed{5 \text{ cm}}$ (**C**) 8 cm (**D**) 10 cm (**E**) None of these.

15. The number of values satisfying the equation $\dfrac{2x^2 - 10x}{x^2 - 5x} = x - 3$ is

(**A**) $\boxed{0}$ (**B**) 1 (**C**) 2 (**D**) 3 (**E**) None of these

16. To which of the following expressions is $\sqrt[8]{x^8} + \sqrt[7]{x^7}$ equal to for all real negative values of x?

(**A**) x (**B**) $2x$ (**C**) $2x^2$ (**D**) $\boxed{0}$ (**E**) x^2

17. Find the sum of all solutions (both real and complex) to the equation $ix^2 + 7x - 12i = 0$, where $i = \sqrt{-1}$.

(**A**) $-7i$ (**B**) $-i$ (**C**) $\boxed{7i}$ (**D**) $8i$ (**E**) None of these

18. Find the product of all solutions of the equation $3^{2x+2} + 3 = 28(3^x)$.

(**A**) $\boxed{-2}$ (**B**) $\frac{1}{3}$ (**C**) $\frac{1}{2}$ (**D**) 3 (**E**) None of these

19. What is the largest solution of the equation $2x^3 - 5x^2 = 8x - 20$?

(**A**) 2 (**B**) $\frac{7}{3}$ (**C**) 4 (**D**) $\frac{9}{2}$ (**E**) $\boxed{\text{None of these}}$

20. The number of square feet in the total surface area of a right circular cylinder is equal to the number of cubic feet in its volume. If the radius of its base is five times its altitude, what is the length of the radius?

(**A**) $\boxed{12}$ (**B**) 8 (**C**) 2 (**D**) 5 (**E**) None of these.

21. On an island, 99% of the population are natives. Some natives emigrate so that only 98% of the population are natives. If the initial population of the island was 1000, how many natives emigrated?

(**A**) 10 (**B**) 20 (**C**) 100 (**D**) $\boxed{500}$ (**E**) None of these

22. If the measures of an angle and its complement are in ratio of 2:7, what is the measure of the angle's supplement, in degrees?

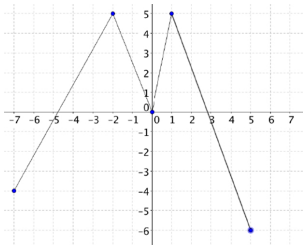
(**A**) $150°$ (**B**) $\boxed{160°}$ (**C**) $70°$ (**D**) $90°$ (**E**) None of these.

23. Find the value of k so that the function $f(x)$ is continuous.

$$f(x) = \begin{cases} kx^2 + 3 & x < 2 \\ k & x \geq 2 \end{cases}$$

(**A**) 2 (**B**) $-\frac{3}{4}$ (**C**) $-\frac{1}{4}$ (**D**) 1 (**E**) $\boxed{\text{None of these}}$

24. The graph of the function f is shown below. How many solutions does the equation $f(f(x)) = 5$ have?



(**A**) 2 (**B**) 3 (**C**) 4 (**D**) $\boxed{6}$ (**E**) None of these

25. Find the inverse function $f^{-1}(x)$ of the function $f(x) = \dfrac{x+2}{x+3}$.

(**A**) $f^{-1}(x) = \frac{3x-2}{x+2}$ (**B**) $f^{-1}(x) = \frac{-1}{x-1}$

(**C**) $\boxed{f^{-1}(x) = \frac{2-3x}{x-1}}$ (**D**) $f^{-1}(x) = \frac{x+3}{x+2}$ (**E**) $f^{-1}(x) = \frac{6+3x}{6+2x}$

26. The equation $2x^{5/6} + 2x^{1/2} = 5x^{2/3}$ has two nonzero real solutions, a and b. Assuming $a < b$, find $\frac{1}{a} + b$.

(**A**) 4 (**B**) 16 (**C**) 64 (**D**) $\boxed{128}$ (**E**) None of these

27. Find the height of a pyramid with a square base whose area is 16 square inches and whose lateral edges have length 5 inches.

(**A**) 4 (**B**) $\boxed{\sqrt{17}}$ (**C**) $\sqrt{18}$ (**D**) 4.5 (**E**) None of these.

28. The student council is made up of four sophomores, two juniors and three seniors. A yearbook photographer would like to line up all members of the student council in a line for a picture. How many different pictures are possible if students in the same grade stand beside each other?

(**A**) $\boxed{4!2!3!3!}$ (**B**) $3!$ (**C**) $4!2!3!$ (**D**) $9!$ (**E**) None of these

29. Find the product of the two smallest solutions of $\cos 5\theta = 0$ on $[0, 2\pi)$.

(**A**) $\frac{3\pi^2}{4}$ (**B**) 0 (**C**) $\boxed{\frac{3\pi^2}{100}}$ (**D**) $\frac{75\pi^2}{4}$ (**E**) None of these

30. If the area of a square inscribed in a circle is 30 square centimeters, what is the perimeter of a square inscribed in a semicircle of the same circle?

(**A**) $\boxed{8\sqrt{3}}$ (**B**) $2\sqrt{3}$ (**C**) $4\sqrt{3}$ (**D**) $6\sqrt{3}$ (**E**) None of these.

31. The area of a given rhombus is 154 u^2. If one of the diagonals has length 14 u, what is the length of the other diagonal?

(**A**) 11 u (**B**) 16 u (**C**) 18 u (**D**) $\boxed{22 \text{ u}}$ (**E**) None of these.

32. Define an operation # on pairs of real numbers as

$$(x_1, y_1)\#(x_2, y_2) = (x_1^2 x_2^2, y_1 y_2).$$

Which of the following could **not** equal $(x_1, y_1)\#(x_2, y_2)$ for any real numbers x_1, y_1, x_2, y_2?

(**A**) $(0, \sqrt{2})$ (**B**) $\boxed{(-1, 0)}$ (**C**) $(1, 4)$ (**D**) $(\pi, -7)$ (**E**) None of these

33. Find the exact value of $\cos 1^\circ + \cos 2^\circ + ... + \cos 179^\circ + \cos 180^\circ$.

(**A**) 1 (**B**) 0 (**C**) 2 (**D**) $\boxed{-1}$ (**E**) None of these

34. A line through the points $(M, -9)$ and $(7, M)$ has slope M. What is the equation of the line that goes through these two points?

(**A**) $y = 6x - 18$ (**B**) $y = 6x + 18$ (**C**) $y = 3x - 24$ (**D**) $y = 6x + 24$ (**E**) $\boxed{y = 3x - 18}$

35. Find the product of all real solutions to the equation $x^{\log_{10} x} = 10$.

(**A**) 10 (**B**) 5 (**C**) $\boxed{1}$ (**D**) $\sqrt{10}$ (**E**) None of these

36. Let (a, b) be a point on the graph of $16x^2 + 36y^2 - 8x + 48y = 127$. What is the largest such b?

(**A**) $\frac{13}{4}$ (**B**) $\frac{8}{3}$ (**C**) $\boxed{\frac{4}{3}}$ (**D**) $\frac{5}{4}$ (**E**) None of these

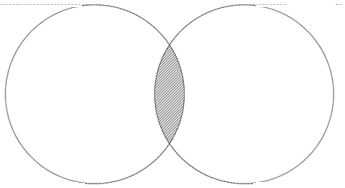
37. Let a and d be real numbers such that $ax^3 + 6x^2 + 9x + d = (f(x))^3$ for some polynomial function $f(x)$. What is the value of ad?

(**A**) $\boxed{6}$ (**B**) 1 (**C**) 54 (**D**) 8 (**E**) None of these

38. Find the horizontal asymptote of the graph of the function $f(x) = \dfrac{-6x^2 + 5x}{4x + 5} + \dfrac{12x^3 + 7x + 1}{8x^2 + 6x - 5}$.

(**A**) $\boxed{y = 2}$ (**B**) $y = 0$ (**C**) $y = \frac{1}{2}$ (**D**) $y = -\frac{5}{4}$ (**E**) None of these

39. In the figure shown, each of the circles has a diameter of 12 and the distance between the intersection points is 6. Find the area of the overlapping region.



(**A**) $8\pi - 6\sqrt{3}$ (**B**) $10\pi - 16\sqrt{3}$ (**C**) $\boxed{12\pi - 18\sqrt{3}}$

(**D**) $6\pi - 9\sqrt{3}$ (**E**) None of these.

40. In the senior class at a particular high school, 45 students are taking calculus, 52 students are taking physics, and 21 students are taking both calculus and physics. If there are 200 people in the senior class, what is the probability that a randomly selected student is taking calculus or physics?

(**A**) 0.59 (**B**) 0.97 (**C**) 0.485 (**D**) $\boxed{0.38}$ (**E**) None of these

41. Find the sum of all solutions to the equation

$$(x^3 + x^2 + 5x - 11)^2 - (4x^2 - 4x + 16)^2 = 0.$$

(A) $\boxed{-2}$ **(B)** 0 **(C)** -8 **(D)** 5 **(E)** None of these

42. Given that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\angle A \cong \angle D$, we can conclude that $\triangle ABC$ and $\triangle DEF$ are

(A) similar. **(B)** congruent. **(C)** both isosceles.

(D) both acute. **(E)** $\boxed{\text{Not enough information is given to draw a conclusion.}}$

43. How many real solutions are there to the equation $\cos^2 x + \sin x = 1$ on $[0, 2\pi)$?

(A) 2 **(B)** $\boxed{3}$ **(C)** 4 **(D)** 5 **(E)** None of these

44. Given a circle centered at (3,4) that passes through (7,1), which of the following is the equation of the tangent line to the circle at the point (7,1)?

(A) $3x - 4y = 17$ **(B)** $4x + 3y = 31$ **(C)** $3x + 4y = 25$ **(D)** $\boxed{4x - 3y = 25}$ **(E)** $4x - 3y = 17$

45. The parabola with the equation $y = ax^2+bx+c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $y = dx^2+ex+f$. Which of the following equal $a+b+c+d+e+f$?

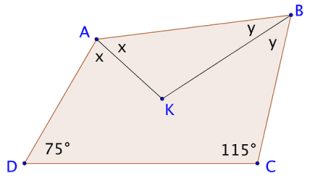
(A) $2b$ **(B)** $2c$ **(C)** $2a + 2b$ **(D)** $2h$ **(E)** $\boxed{2k}$

46. At a Presidents' Day sale, you buy a scarf regularly priced at \$20. You also buy a sweater without a price tag which is on a 50% off rack. After using a 10% off your total purchase coupon, you pay \$45. Assuming there is no tax, what was the original price of the sweater?

(A) \$67.50 **(B)** $\boxed{\$60.00}$ **(C)** \$55.56 **(D)** \$50.00 **(E)** None of these

47. In the diagram shown, $m\angle D = 75°$ and $m\angle C = 115°$. There is a point K interior to quadrilateral $ABCD$ such that $\overline{AK}$ bisects angle $\angle A$ and $\overline{BK}$ bisects $\angle B$. Find the measure of $\angle AKB$.

(A) 75° **(B)** $\boxed{95°}$ **(C)** 105° **(D)** 110° **(E)** None of these.



48. A box contains 1 white, 3 purple, and 2 gold balls. A second box contains 2 purple and 2 gold balls. One ball is selected at random from each box. What is the probability they are the same color?

(A) $\boxed{\frac{5}{12}}$ **(B)** $\frac{5}{6}$ **(C)** $\frac{16}{45}$ **(D)** $\frac{1}{2}$ **(E)** None of these

49. The graph of the function $y = \dfrac{x^2 - 5x + 6}{2x^2 + 3x - 4}$ has two vertical asymptotes $x = a$ and $x = b$. Find ab.

(A) $\boxed{-2}$ **(B)** 6 **(C)** -6 **(D)** -4 **(E)** None of these

50. Find the product of all real numbers k for which the function $f(x) = kx^2 + x + k$ touches, but does not cross, the x-axis.

(**A**) $\frac{1}{2}$ (**B**) 0 (**C**) $\boxed{-\frac{1}{4}}$ (**D**) -1 (**E**) None of these