The 39th Annual

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round: February 29, 2020 at Regional Testing Centers
Second Round: April 25, 2020 at Auburn University at Montgomery

ALGEBRA II WITH TRIGONOMETRY EXAMINATION

Construction of this test directed
by
Ashley Johnson, University of North Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- $\log(x)$ means $\log_{10}(x)$ and $\ln(x)$ means $\log_e(x)$.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then $m\angle A$ is the measure of angle A in degrees.
- If A and B are points on a circle, then $\overset{\frown}{AB}$ is the arc between A and B.
- If A and B are points on a circle, then $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees.
- If $\overline{AB} \cong \overline{CD}$, then $\overline{AB}$ and $\overline{CD}$ are congruent.
- If $\triangle ABC \cong \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are congruent.
- If $\triangle ABC \sim \triangle DEF$, then $\triangle ABC$ and $\triangle DEF$ are similar.
- If ℓ, m are two lines, then $\ell \perp m$ means ℓ and m are perpendicular.

Editing by Miranda Bowie and Ashley Johnson, University of North Alabama
Printing by The University of Alabama at Birmingham

Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Traffic Controller
- Climate Analyst
- Estimator
- Research Scientist
- Computer Programmer
- Cryptanalyst
- Professor
- Pollster
- Population Ecologist
- Operations Research
- Data Mining
- Mathematician
- Meteorologist
- Medical Doctor
- Lawyer
- Actuary
- Statistician

Where can I work? What sorts of companies hire mathematicians? Well just to name a few...

- **U.S. Government Agencies** such as the National Center for Computing Sciences, the National Institute of Standards and Technology (NIST), the National Security Agency (NSA), and the U.S. Department of Energy.
- **Government labs and research offices** such as Air Force Office of Scientific Research, Los Alamos National Laboratory, and Sandia National Laboratory.
- **Engineering research organizations** such as AT&T Laboratories - Research, Exxon Research and Engineering, and IBM Research.
- **Computer information and software firms** such as Adobe, Google, Mentor Graphics, Microsoft, and Yahoo Research.
- **Electronics and computer manufacturers** such as Alcatel-Lucent, Hewlett-Packard, Honeywell, Philips Research, and SGI.
- **Aerospace and transportation equipment manufacturers** such as Boeing, Ford, General Motors, and Lockheed Martin.
- **Transportation service providers** such as FedEx Corporation and United Parcel Service (UPS).
- **Financial service and investment manangement firms** such as Citibank, Morgan Stanley, and Prudential.

A Mathematics Major isn't just for those wanting to be Mathematicians!

- The top scoring major on the Law School Entrance Exam (LSAT) is Mathematics (Source: Journal of Economic Education)
- Mathematics is also a top 5 scoring major on the Medical School Entrance Exam (MCAT) (Source: American Institute of Physics)

Study in the field of mathematics offers an education with an emphasis on careful problem solving, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, and health and environmental fields require mathematical techniques for their solutions. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems.

1. Simplify $\dfrac{8^{\frac{1}{2}}\ 3^{\frac{5}{6}}}{6^{\frac{3}{4}}\ 4^{\frac{1}{3}}}$.

(**A**) $\boxed{6^{\frac{1}{12}}}$ (**B**) $24^{\frac{1}{4}}$ (**C**) $2^{\frac{1}{4}}$ (**D**) $2^{\frac{1}{12}}$ (**E**) None of these

2. Jane picks a number at random between 1 and 8 (inclusive) and Ryan also picks a number between 1 and 8 (inclusive) independent of Jane's selection. What number is the most likely value of the sum of their two numbers?

(**A**) 8 (**B**) $\boxed{9}$ (**C**) 10 (**D**) 11 (**E**) None of these

3. The equation $x^4 = y^2 + 71$ has only one solution (a, b) where both a, b are positive integers. Find $a + b$.

(**A**) 37 (**B**) $\boxed{41}$ (**C**) 48 (**D**) 55 (**E**) None of these

4. Suppose that f is a function for which for all $x > 0$ we have that $f(2x) = \dfrac{2}{2+x}$. What is the value of $f(1)$?

(**A**) $\dfrac{1}{3}$ (**B**) $\dfrac{1}{2}$ (**C**) $\dfrac{2}{3}$ (**D**) $\boxed{\dfrac{4}{5}}$ (**E**) None of these

5. Find the number of solutions to the equation $2^{-|x|} = x^2 + 1$.

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

6. Consider the function $f(x) = x^3 + ax^2 + 5$. If $f(2) > 18$ and $f(-2) < 11$, how many possible integer values are there for a?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) None of these

7. If $f(x) = \dfrac{x}{3x+1}$ and $(f \circ g)(x) = x$, find $g(x)$.

(**A**) $\boxed{g(x) = \dfrac{x}{1-3x}}$ (**B**) $g(x) = \dfrac{3x+1}{x}$

(**C**) $g(x) = \dfrac{1}{3x+1}$ (**D**) $g(x) = \dfrac{1}{1-3x}$ (**E**) $g(x) = \dfrac{1-3x}{x}$

8. Find the sum of the squares of all solutions to the equation $(x-2)(6x+2) = 20$.

(**A**) 8 (**B**) 493 (**C**) $\dfrac{25}{9}$ (**D**) $\boxed{\dfrac{97}{9}}$ (**E**) None of these

9. Let $f(x) = 2x^2 + x + 2$ and $g(x) = \frac{1}{2}x$. What is the minimum vertical distance between the graphs of $f(x)$ and $g(x)$?

(**A**) $\dfrac{61}{32}$ (**B**) $\boxed{\dfrac{63}{32}}$ (**C**) 2 (**D**) $\dfrac{67}{32}$ (**E**) None of these

10. Find the $a+bi$ form of the expansion of $(i+1)^6$, where $i=\sqrt{-1}$.

(**A**) $-12i$ (**B**) $\boxed{-8i}$ (**C**) 0 (**D**) $4\sqrt{2}+4\sqrt{2}i$ (**E**) None of these

11. Find the sum of all real values of x for which $2x^{1/2}+15x^{-1/2}-13=0$.

(**A**) $\frac{13}{2}$ (**B**) $\frac{17}{2}$ (**C**) $\boxed{\frac{109}{4}}$ (**D**) $\frac{169}{4}$ (**E**) None of these

12. A survey of 140 employees at an accounting firm yielded the following information:
62 were under age 40
48 were 40 or older and making \$50,000 or more
40% of the employees making less than \$50,000 were under age 40
Find the probability that a randomly selected employee is under age 40 and making \$50,000 or more.

(**A**) $\boxed{\frac{3}{10}}$ (**B**) $\frac{26}{35}$ (**C**) $\frac{11}{14}$ (**D**) $\frac{17}{140}$ (**E**) None of these

13. Let $f(x)=x^2+2x$. When is $f(\frac{1}{x})=\frac{1}{f(x)}$?

(**A**) Never (**B**) $\boxed{\text{For a single value of } x}$ (**C**) For two values of x

(**D**) For four values of x (**E**) For all values of x

14. How many positive roots does the polynomial $f(x)=3x^6-4x^4-x-2$ have?

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

15. Let x be the smallest positive integer such that x modulo 11 is 7 and x modulo 13 is 8. What is the value of x modulo 12?

(**A**) $\boxed{1}$ (**B**) 3 (**C**) 5 (**D**) 7 (**E**) None of these

16. What is the remainder when $x^{87}+x^{42}-x^{31}+x^5+1$ is divided by x^3+x?

(**A**) $x+1$ (**B**) x^2+x (**C**) x^2+1 (**D**) $\boxed{x^2+x+1}$ (**E**) None of these

17. How many numbers of the form $4a37b$, where both a and b represent single digit numbers, are divisible by 36?

(**A**) $\boxed{2}$ (**B**) 4 (**C**) 8 (**D**) 16 (**E**) None of these

18. The fraction $\frac{2}{5}$ can be written uniquely as $\frac{1}{a}+\frac{1}{b}$ for $a\neq b$ and a,b positive integers. What is $a+b$?

(**A**) 5 (**B**) 10 (**C**) 14 (**D**) $\boxed{18}$ (**E**) None of these

19. The graph of which of the following functions has a horizontal asymptote at $y=2$?

(**A**) $f(x)=\ln(x)+2$ (**B**) $f(x)=e^{x+2}$ (**C**) $\boxed{f(x)=\frac{1}{x}+2}$ (**D**) $f(x)=2^{x-2}$ (**E**) None of these

20. Consider four integers R, S, T, and U. The mean of these integers is 6, their median is 5.5, their mode is 3, and their range is 7. What is the product RSTU of the four integers?

(**A**) 567 (**B**) 693 (**C**) $\boxed{720}$ (**D**) 792 (**E**) None of these

21. How many integers between 2 and 10,001 are perfect squares?

(**A**) $\boxed{99}$ (**B**) 100 (**C**) 101 (**D**) 102 (**E**) None of these

22. How many x-intercepts does the graph of the function $f(x)$ defined as

$$f(x) = \begin{cases} x - 4 & x < -2 \\ x^2 + x & -2 \leq x \leq 0 \\ x^3 + 4 & x > 0 \end{cases}$$

have?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) None of these

23. Let matrices A and B be defined as $A = \begin{bmatrix} 14 & 0 & 18 & -12 \\ 2 & 1 & -7 & 0 \\ 29 & 31 & -14 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 91 & 6 & -22 \\ 30 & -14 & 8 \\ -25 & -2 & 17 \\ 42 & 19 & 20 \end{bmatrix}$.
Find the entry in the 2nd row, 2nd column of the product AB.

(**A**) -14 (**B**) 0 (**C**) 56 (**D**) 234 (**E**) $\boxed{\text{None of these}}$

24. An unfair coin which comes up heads with a probability of 0.80 is flipped 10 times. What is the probability it comes up heads exactly once?

(**A**) $\frac{1}{10}$ (**B**) $\frac{4}{5}$ (**C**) $\frac{4}{5^{10}}$ (**D**) $\boxed{\frac{8}{5^9}}$ (**E**) None of these

25. Let $f(x) = 3x$ and $g(x) = \frac{1}{x} - 3$. Find the value of $\left[(f^{-1} \circ g^{-1})(-1)\right]^{-1}$.

(**A**) -4 (**B**) $-\frac{3}{4}$ (**C**) $\boxed{6}$ (**D**) 12 (**E**) None of these

26. The three-digit numbers acb, $a79$, $b0c$, and $bb1$ are consecutive terms in an arithmetic sequence, where a, b, and c each represent a single digit. How many possible values are there for a?

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) None of these

27. Which of the following (x, y) pairs is a solution to the equation $x^4 + 2x^2y^2 + 9y^4 = 0$?

(**A**) $(-179 + 179\sqrt{3}\,i, 179)$ (**B**) $(179, -179 + 179\sqrt{2}\,i)$ (**C**) $(179 - 179\sqrt{3}\,i, 179)$

(**D**) $(179, -179 + 179\sqrt{3}\,i)$ (**E**) $\boxed{(-179 + 179\sqrt{2}\,i, 179)}$

28. Let $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ be the four corners of a square in the plane. Find the sum of all values of m for which the line $y = mx$ divides the square into two regions, one with area 0.6 and one with area 0.4.

(**A**) $\frac{8}{5}$ (**B**) $\frac{9}{4}$ (**C**) $\boxed{\frac{41}{20}}$ (**D**) $\frac{49}{20}$ (**E**) None of these

29. The difference quotient of a function $f(x)$ is defined as $\frac{f(x+h)-f(x)}{h}$ for $h \neq 0$. Evaluate and simplify the difference quotient for $f(x) = x^3 + 6x$.

(**A**) $3x^2 + xh + h^2 + 6$ (**B**) 1 (**C**) $\boxed{3x^2 + 3xh + h^2 + 6}$ (**D**) $h^2 + 6$ (**E**) $3x^2 + 6xh + 6$

30. The ceiling of a number x, denoted as $\lceil x \rceil$, is the smallest integer that is greater than or equal to x. The floor of a number x, denoted as $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. For example, $\lceil 1.1 \rceil = 2$ and $\lfloor 1.1 \rfloor = 1$. Find $\lceil -2.4 \rceil + \lfloor 8.9 \rfloor$.

(**A**) 5 (**B**) $\boxed{6}$ (**C**) 7 (**D**) 8 (**E**) None of these

31. Which of the following numbers is the largest?

$$\frac{2063}{4050} \quad \frac{14}{27} \quad \frac{4}{3\pi} \quad \frac{\log_{10}(96)}{4} \quad \frac{7}{\sqrt{165}}$$

(**A**) $\frac{2063}{4050}$ (**B**) $\frac{14}{27}$ (**C**) $\frac{4}{3\pi}$ (**D**) $\frac{\log_{10}(96)}{4}$ (**E**) $\boxed{\frac{7}{\sqrt{165}}}$

32. You are traveling 65 miles per hour in a car on a straight stretch of road parallel to a train traveling in the same direction. If it takes you 6 minutes to pass the train, which is 1.25 miles in length, how fast is the train traveling in miles per hour?

(**A**) 48.2 (**B**) $\boxed{52.5}$ (**C**) 58.6 (**D**) 64.8 (**E**) None of these

33. Find the distance between the two points of intersection of the circle given by $(x-3)^2 + (y+5)^2 = 9$ and the line $x + y = 1$.

(**A**) $\boxed{3\sqrt{2}}$ (**B**) $4\sqrt{2}$ (**C**) $3\sqrt{6}$ (**D**) 6 (**E**) None of these

34. Let $f(x) = \log_2(x)$. Which of the following is equal to $f(x^2) - f(4x) + f(8)$?

(**A**) $f(x) - 5$ (**B**) $3f(x) + 1$ (**C**) $[f(x)]^2 - 4f(x) + 3$ (**D**) $\boxed{f(x) + 1}$ (**E**) $f(2x - 4)$

35. Let $D = a^2 + b^2 + (ab)^2$, where a, b are consecutive integers. Then $\sqrt{D}$ is:

(**A**) always an even integer (**B**) sometimes an odd integer, sometimes not

(**C**) $\boxed{\text{always an odd integer}}$ (**D**) sometimes rational, sometimes not (**E**) always irrational

36. The solutions of the equation $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. What is the median of the solutions?

(**A**) $\frac{4}{15}$ (**B**) $\frac{4}{5}$ (**C**) $\frac{3}{8}$ (**D**) $\boxed{\frac{3}{4}}$ (**E**) None of these

37. Define an operation $*$ on ordered pairs of integers as $(a,b)*(c,d) = (a+c, b^2-d^2)$. Find $(-2,4)*(0,1)$.

(**A**) $\boxed{(-2,15)}$ (**B**) $(-2,9)$ (**C**) $(2,15)$ (**D**) $(2,9)$ (**E**) None of these

38. The number $x = \sqrt[3]{9+\sqrt{17}} + \sqrt[3]{9-\sqrt{17}}$ is a root of which of the following polynomials?

(**A**) $x^3+12x+18$ (**B**) $x^3-12x+18$ (**C**) $\boxed{x^3-12x-18}$ (**D**) x^3-18 (**E**) x^3-12x

39. Define a sequence recursively as $a_1 = 3$ and $a_n = 2a_{n-1} - 1$ for $n \geq 2$. Find a_{10}.

(**A**) 19 (**B**) 17 (**C**) $\boxed{1025}$ (**D**) 2020 (**E**) None of these

40. A two-digit number is multiplied by the number obtained by reversing its digits. What is the largest possible value for this product if the product of the two digits is 12?

(**A**) 1462 (**B**) $\boxed{1612}$ (**C**) 3627 (**D**) 4032 (**E**) None of these

41. A man planning to walk 40 miles realizes that by walking one mile per hour faster, he could make the journey in 2 hours less time. How many miles per hour was he originally going to be walking?

(**A**) 3 (**B**) 3.5 (**C**) $\boxed{4}$ (**D**) 4.5 (**E**) None of these

42. What is the largest integer value of k for which the equation $kx^2 + 4\sqrt{3}x + k = 1$ has at least one real solution?

(**A**) -3 (**B**) 0 (**C**) 3 (**D**) $\boxed{4}$ (**E**) None of these

43. If $f(x) = ax^2 + bx + c$ is a quadratic function with vertex at $(2,4)$, going through point $(1,3)$, what is $a+b+c$?

(**A**) $\boxed{3}$ (**B**) 4 (**C**) 6 (**D**) 7 (**E**) None of these

44. Currently, Derrick and Sonya's ages are in the ratio of 3:4. Five years from now, their ages will be in the ratio of 4:5. What is the sum of their current ages?

(**A**) 28 (**B**) $\boxed{35}$ (**C**) 42 (**D**) 49 (**E**) None of these

45. The graph of a function $g(x)$ is symmetric about the y-axis. What must be true about $g(x)$?

(**A**) $\boxed{g(x) \text{ is an even function}}$ (**B**) $g(x)$ is an odd function

(**C**) $g(x)$ has an even number of roots (**D**) $g(x)$ has an odd number of roots (**E**) None of these

46. Find the smallest solution to the equation $\dfrac{x^2-8x-3}{x^3-7x} = 2$.

(**A**) $\sqrt{2}$ (**B**) $\sqrt{3}$ (**C**) $\dfrac{1}{2}$ (**D**) $\dfrac{3}{2}$ (**E**) $\boxed{\text{None of these}}$

47. Let $h(x) = 2x^5 - 19x^4 - 12x^3 - 4x^2 - x - 90$. For large, positive values of x, the value of $h(x)$ is approaching

(**A**) negative infinity (**B**) -90 (**C**) 0 (**D**) 2 (**E**) $\boxed{\text{infinity}}$

48. How many different arrangements are there of the letters in the word PUPPIES?

(**A**) 35 (**B**) 120 (**C**) $\boxed{840}$ (**D**) 5040 (**E**) None of these

49. Find the product of all solutions to the equation $|2x + 3| = 5 - x$.

(**A**) $\boxed{-\frac{16}{3}}$ (**B**) $-\frac{4}{9}$ (**C**) $\frac{4}{3}$ (**D**) $\frac{16}{9}$ (**E**) None of these

50. Find the y-coordinate of the y-intercept of a line with an x-intercept of $(3, 0)$, and a slope of $-\frac{5}{7}$.

(**A**) $-\frac{7}{15}$ (**B**) $\frac{7}{15}$ (**C**) $-\frac{15}{7}$ (**D**) $\boxed{\frac{15}{7}}$ (**E**) None of these